

MA366 Make-up Final

**Last Name:**.....

**First Name:**.....

**Show all work.** A correct answer without supporting work is worth NO credit! (Some calculators can solve differential equations.) There should be no “hard” integrals, unless you mess up somewhere. If this happens, just leave it as an integral and explain how to finish the problem.

- (1) The following vectors  $X_1$  and  $Y_1$  are eigenvectors for a certain  $3 \times 3$  matrix  $A$  corresponding to the eigenvalues 3 and  $1 + 2i$  respectively. Find the general solution to the system  $X' = AX$  **in real form**. No complex numbers allowed!

5 pts.

$$X_1 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} i - 1 \\ i \\ -2 \end{bmatrix}.$$

- (2) Given that  $X_1$ ,  $X_2$  and  $X_3$  are eigenvectors for the following matrix, find the general solution to  $X' = AX$ . *Hint:* To find the eigenvalue, compute  $AX_i$ . 5 pts.

$$A = \begin{bmatrix} -3 & 6 & -3 \\ 4 & -3 & 2 \\ 6 & -12 & 6 \end{bmatrix} \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \quad X_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \quad X_3 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

- (3) The characteristic polynomial for the following matrix is  $p(r) = -(r - 3)^2(r - 1)$  and the vector  $Y_1$  is an eigenvector corresponding to  $r = 1$ . Find the general solution to the system  $X' = AX$ .

10 pts.

$$\begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} Y_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

(4) For the following matrix  $A$ :

10 pts.

(a) Find  $e^{tA}$ .

(b) Find the general solution to  $X' = AX$ .

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

- (5) A certain  $6 \times 6$  matrix  $A$  has characteristic polynomial  $p(r) = -(r-2)^3(r-5)^3$ . Let  $X$  be a generalized eigenvector for  $A$  corresponding to  $r = 2$ . Give a formula for  $e^{tA}X$  that does not require summing an infinite series. Your formula should use as few matrix products as possible relative to the given information.

5 pts.

- (6) Find all singular points for the following differential equation and state which are regular. **Don't forget to justify your answers!**

6 pts.

$$x(x-1)^3y'' + xy' + (x-1)y = 0$$

- (7) Substitute  $y = \sum_{n=0}^{\infty} a_n x^n$  into the differential equation
- $$x^2 y'' + (x^3 - 2)y = 0$$

and simplify until you obtain an expression of the form

$$\sum_{n=?}^{\infty} ?x^n + \sum_{n=?}^{\infty} ?x^n + \sum_{n=?}^{\infty} ?x^n = 0$$

where the exponent of  $x$  in each sum is  $n$  and the question marks are explicit expressions. **Do not simplify further!** 10 pts.

10 pts.

- (8) In attempting to solve a certain differential equation, we substituted  $y = \sum_{n=0}^{\infty} a_n x^n$  into the differential equation and simplified, obtaining

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

- (a) Continue the solution process to obtain the recursion relation.
- (b) Find the first three **non-zero** terms of the power series expansion for the solution  $y_1$  satisfying  $y_1(0) = 1, y_1'(0) = 2$ .

- (9) The following differential equation has a regular singularity at  $x = 0$ . 9 pts.

$$x^2(x^2 + 1)y'' + x(x^3 + 3)y' + (x + 1)y = 0.$$

- (a) Give the approximating Euler equation.
- (b) Give the indicial equation.
- (c) Use Theorem 5.6.1 on p. 289 of the text to describe the expected form of the solutions. **Do not find the coefficients of the series expansions!**

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- (10) You are given that  $y(x) = x^{-2}$  is a solution of the following differential equation. Use the method of reduction of order to find a second independent solution. **Other methods will not receive credit!**

10 pts.

$$x^2 y'' - 6y = 0.$$

- (11) Use the definition of the Laplace transform to compute the Laplace transform  $\mathcal{L}(f)$  of the function  $f(t)$  defined below.

**Other methods will not receive credit!**

4 pts.

$$f(t) = \begin{cases} 5, & 0 \leq t < 2 \\ e^{-3t}, & 2 \leq t. \end{cases}$$

4 pts.

(12) Find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2 + 5s + 6}.$$

4 pts.

(13) Find the inverse Laplace transform of

$$F(s) = \frac{e^{-3s}}{s^2 + 5s + 6}.$$

(14) Find the inverse Laplace transform of

4 pts.

$$F(s) = \frac{s + 1}{s^2 + 2s + 10}$$

(15) Find the Laplace transform  $Y(s)$  of the solution  $y(t)$  to the following initial value problem in terms of  $a$  and  $b$ . **Do not find  $y(t)$ . All we want is  $Y(s)$ !**

4 pts.

$$y'' + 7y' - 3y = e^{3t}, \quad y(0) = a, \quad y'(0) = b.$$