100

8 pts. (1) Given that X_1, X_2 and X_3 are eigenvectors for the following matrix, find the general solution to $X' = AX$.

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} X_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} X_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$A\xi = r\xi$

1) $AX_1 = r_1 X_1$ $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+4-2 \\ -3+4 \\ -3+1+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = r_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $r_1 = 1$

2) $AX_2 = r_2 X_2$ $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2+12-6 \\ -6+12 \\ -6+3+9 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = r_2\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ $r_2 = 2$

3) $AX_3 = r_3 X_3$ $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1+12-8 \\ -3+12 \\ -3+3+12 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 12 \end{bmatrix} = r_3\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ $r_3 = 3$

$X = c_1 e^{r_1 t} X_1 + c_2 e^{r_2 t} X_2 + c_3 e^{r_3 t} X_3$

$$\boxed{X(t) = c_1 e^{t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}}$$

100

(2) The characteristic polynomial for the following matrix is $p(r) = -(r-5)^2(r-7)$ and the vector Y_1 is an eigenvector corresponding to $r = 7$. Find the general solution to the system $X' = AX$. 14 pts.

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 2 & 3 & 3 \\ 4 & -4 & 9 \end{bmatrix} \quad Y_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$4 \quad -4 \quad 4$

$-4+12 \quad 4-12 \quad 2-6+12$

$-8+16 \quad 8-16 \quad 4-12+16$

$$B = A - 5I = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -2 & 3 \\ 4 & -4 & 4 \end{bmatrix} \qquad B^2 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -2 & 3 \\ 4 & -4 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 2 & -2 & 3 \\ 4 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 4 \\ 8 & -8 & 8 \\ 8 & -8 & 8 \end{bmatrix}$$

$4-12+16$

$2-6+12$

$-4+12$

$4 \quad -12$

$$A = B + 5I$$

$$\rightarrow \begin{bmatrix} 4 & -4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad 4x - 4y + 4z = 0$$

$$x = y - z \qquad y = t$$

$$B^2 X = 0 \qquad x = t - s \qquad z = s$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t-s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\xi_1 \qquad \xi_2$

$$e^{tA}X = e^{t(B+5I)}X = e^{5t}\left(I + Bt + \cancel{\frac{t^2B^2}{2!}}^{\,0} + \cdots\right)X$$

$$= e^{5t}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t \\ 2t & -2t & 3t \\ 4t & -4t & 4t \end{bmatrix}\right)X = e^{5t}\begin{bmatrix} 1 & 0 & t \\ 2t & 1-2t & 3t \\ 4t & -4t & 1+4t \end{bmatrix}X$$

$$X_1 = e^{5t}\begin{bmatrix} 1 & 0 & t \\ 2t & 1-2t & 3t \\ 4t & -4t & 1+4t \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = e^{5t}\begin{bmatrix} 1 \\ 2t+1-2t \\ 4t-4t \end{bmatrix} = e^{5t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$X_2 = e^{5t}\begin{bmatrix} 1 & 0 & t \\ 2t & 1-2t & 3t \\ 4t & -4t & 1+4t \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = e^{5t}\begin{bmatrix} -1+t \\ -2t+3t \\ -4t+1+4t \end{bmatrix} = e^{5t}\begin{bmatrix} -1+t \\ t \\ 1 \end{bmatrix}$$

$$\boxed{X(t) = C_1e^{5t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + C_2e^{5t}\begin{bmatrix} -1+t \\ t \\ 1 \end{bmatrix} + C_3e^{7t}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}$$

100

(3) The following matrix A has characteristic polynomial $p(r) = -(r-2)^3$.

$$\begin{bmatrix} 3 & -1 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

10 pts. (a) Find e^{tA}.

$1-1 \quad -1-1+2 \quad 1+1-2$
$1+1 \quad -1+1-2 \quad 1-1+2$
$2 \quad 2-4 \quad 2+4$

$B = A - 2I = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}$

$A = B + 2I$

$B^2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 2 \\ 2 & -2 & 2 \end{bmatrix}$

$B^3 X = 0$

$B^3 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 2 \\ 2 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$e^{2tI} = e^{2t}$

$e^{tA}X = e^{t(B+2I)}X = e^{2t}e^{Bt}X = e^{2t}\left(I + tB + \frac{t^2B^2}{2!} + \cancel{\frac{t^3B^3}{3!}}^{\,0} + \cdots\right)X$

$2-2 \quad -2-2+4 \quad 2+2-4$
$2-2 \quad -2-2+4 \quad 2+2-4$

$e^{2t}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 2 \\ 2 & -2 & 2 \end{bmatrix}\right)X = e^{2t}\begin{bmatrix} 1+t & -t & t \\ t+t^2 & 1+t-t^2 & -t+t^2 \\ t^2 & 2t-t^2 & 1-2t+t^2 \end{bmatrix}X$

$X = c_1 e^{2t}\begin{bmatrix} 1+t \\ t+t^2 \\ t^2 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -t \\ 1+t-t^2 \\ 2t-t^2 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} t \\ -t+t^2 \\ 1-2t+t^2 \end{bmatrix}$

$$e^{tA} = e^{2t}\begin{bmatrix} 1+t & -t & t \\ t+t^2 & 1+t-t^2 & -t+t^2 \\ t^2 & 2t-t^2 & 1-2t+t^2 \end{bmatrix}$$

1 60

(b) Find the solution to $X' = AX$ which satisfies the initial condition 4 pts.

$$X(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$X = c_1 e^{2t}\begin{bmatrix} 1+t \\ t+t^2 \\ t^2 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -t \\ 1+t-t^2 \\ 2t-t^2 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} t \\ -t+t^2 \\ 1-2t+t^2 \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = c_1 e^{0}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{0}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{0}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$c_1 = 1 \qquad c_2 = 1 \qquad c_3 = 1$

$$X(t) = e^{2t}\begin{bmatrix} (1+t & -t & t) \\ (t+t^2 & 1+t-t^2 & -t+t^2) \\ (t^2 & 2t-t^2 & 1-2t+t^2) \end{bmatrix}$$

$$\boxed{X(t) = e^{2t}\begin{bmatrix} 1+t \\ 1+t+t^2 \\ 1+t^2 \end{bmatrix}}$$ ✓

100

8 pts.

(4) The following vectors X_1 and Y_1 are eigenvectors for a certain 3×3 matrix A corresponding to the eigenvalues $1 - 2i$ and 3 respectively. Find the general solution to the system $X' = AX$ **in real form**. No complex numbers allowed!

$$X_1 = \begin{bmatrix} -1 \\ i+1 \\ 3i \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$X = C_1 e^{r_1 t} X_1 + C_2 e^{r_2 t} Y_1 \qquad C_2 e^{r_2 t} Y_1 = C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C_1 e^{r_1 t} X_1 = C_1 e^{(1-2i)t} \begin{bmatrix} -1 \\ i+1 \\ 3i \end{bmatrix} = C_1 e^{t} (\cos 2t - i \sin 2t) \begin{bmatrix} -1 \\ i+1 \\ 3i \end{bmatrix}$$

$-i^2 \sin 2t$
$+(+1)\sin 2t$
$-3i^2 \sin 2t$

$$= C_1 e^{t} \begin{bmatrix} -\cos 2t + i \sin 2t \\ i\cos 2t + \cos 2t + \sin 2t - i \sin 2t \\ 3i\cos 2t + 3\sin 2t \end{bmatrix} = C_1 e^{t} \begin{bmatrix} -\cos 2t \\ \cos 2t + \sin 2t \\ 3\sin 2t \end{bmatrix} + i C_2 e^{t} \begin{bmatrix} +\sin 2t \\ \cos 2t - \sin 2t \\ 3\cos 2t \end{bmatrix}$$

ξ_1 ↗ $\qquad \xi_2$ ↗

$$\boxed{X(t) = C_1 e^{t} \begin{bmatrix} -\cos 2t \\ \cos 2t + \sin 2t \\ 3\sin 2t \end{bmatrix} + C_2 e^{t} \begin{bmatrix} \sin 2t \\ \cos 2t - \sin 2t \\ 3\cos 2t \end{bmatrix} + C_3 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

100

(5) A certain ~~7×7~~ 9x9 matrix A has characteristic polynomial $p(r) = -(r-3)^5(r+1)^4$. Let X be a generalized eigenvector for A corresponding to $r = -1$. Give a formula for $e^{tA}X$ that does not require summing an infinite series. Your formula should use as few matrix products as possible relative to the given information. 6 pts.

$B = A - (-1)I = A + I \qquad B^4 X = 0$

$A = B - I$

$e^{tA}X = e^{t(B-I)}X = e^{-t}e^{Bt}X = e^{-t}\left(I + Bt + \frac{t^2B^2}{2!} + \frac{t^3B^3}{3!} + \cancel{\frac{t^4B^4}{4!}} + \cdots\right)X$ 0

$e^{tA}X = e^{-t}\left(I + Bt + \frac{t^2B^2}{2!} + \frac{t^3B^3}{3!}\right)X$

$$\boxed{e^{tA}X = e^{-t}\left(X + tBX + \frac{t^2B^2}{2!}X + \frac{t^3B^3}{3!}X\right)}$$

(6) The function

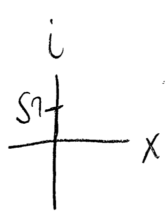
$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

solves the differential equation

$$(x^2+7)y'' + (x+5)y' + (x^2-9)y = 0.$$

What is the largest r for which you can be certain that the series converges for all x satisfying $|x| < r$. 4 pts.

$y'' + \frac{(x+5)}{x^2+7}y' + \frac{x^2-9}{x^2+7}y = 0$



$x^2 + 7 = 0$

$x^2 = -7$

$x = \pm\sqrt{7}\,i$

$\boxed{r = \sqrt{7}}$

100

(7) Substitute $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation

$$(x^2+2)y'' + xy = 0$$

and simplify until you obtain an expression of the form

$$\sum_{n=?}^{\infty} ?x^n + \sum_{n=?}^{\infty} ?x^n + \sum_{n=?}^{\infty} ?x^n + \sum_{n=?}^{\infty} ?x^n = 0$$

where the exponent of x in each sum is n and the question marks are explicit expressions. (You do not need to use exactly 4 summation signs.) **Do not simplify further!**

8 pts.

$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=0} n(n-1) a_n x^{n-2}$

$x^2 y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^n$

$2y'' = \sum_{n=0}^{\infty} 2n(n-1) a_n x^{n-2}$

$k = n-2$
$n = k+2 \quad \Rightarrow \sum_{n=-2}^{\infty} 2(n+2)(n+1) a_{n+2} x^n$

$xy = \sum_{n=0}^{\infty} a_n x^{n+1}$

$k = n+1$
$n = k-1 \quad \Rightarrow \sum_{n=1}^{\infty} a_{n-1} x^n$

$$\boxed{\sum_{n=0}^{\infty} n(n-1) a_n x^n + \sum_{n=-2}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0}$$

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(8) In attempting to solve a certain differential equation, we substituted $y = \sum_{n=0}^{\infty} a_n x^n$ into the differential equation and simplified, obtaining

10 pts.

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=1}^{\infty} 6a_{n-1}x^n = 0.$$

Given that $y(0) = 1$ and $y'(0) = 2$, find a_0, a_1, a_2 , a_3, a_4, and a_5 .

$y(0) = a_0 = 1 \quad y'(0) = a_1 = 2$

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n + \sum_{n=1}^{\infty} 6a_{n-1}x^n = 0$$

$$(1)(2)a_2 x^0 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+2} + 6a_{n-1}\right]x^n = 0$$

$2a_2 = 0$

$a_2 = 0$

$(n+1)(n+2)a_{n+2} + 6a_{n-1} = 0 \quad n \geq 1$

$a_{n+2} = \dfrac{-6a_{n-1}}{(n+1)(n+2)} \quad n \geq 1$

$n=1 \quad a_3 = \dfrac{-6a_0}{2\cdot 3} = \dfrac{-6}{6} = -1$

$n=2 \quad a_4 = \dfrac{-6a_1}{3\cdot 4} = \dfrac{-6(2)}{3\cdot 4} = \dfrac{-12}{12} = -1$

$n=3 \quad a_5 = \dfrac{-6a_2}{4\cdot 5} = \dfrac{-6(0)}{4\cdot 5} = 0$

$$\boxed{\begin{aligned} a_0 &= 1 \\ a_1 &= 2 \\ a_2 &= 0 \\ a_3 &= -1 \\ a_4 &= -1 \\ a_5 &= 0 \end{aligned}}$$

(9) Consider the following differential eauation.

$$(x^3 + 3x^2)y'' + (x^3 + 2x^2 + x)y' + x^2y = 0.$$

$x^2(x+3)y'' + x(x^2+2x+1)y' + x^2y = 0$

4 pts. (a) Show that this equation has a regular singularity at $x = 0$.

$p(x) = \frac{x^2+2x+1}{x(x+3)}$

$q(x) = \frac{1}{x+3}$

$y'' + \frac{x^3+2x^2+x}{x^3+3x^2}y' + \frac{x^2}{x^3+3x^2}y = 0$

$y'' + \frac{\cancel{x}(x^2+2x+1)}{x^{\cancel{2}}(x+3)}y' + \frac{\cancel{x^2}}{\cancel{x^2}(x+3)}y = 0$

$x = 0$ is a singularity because it makes the denominator of $p(x) = 0$

$x^2P(x) = x^2(x+3)$

$P(x) = x+3$

$P(0) = 3 \neq 0$ ✓

$x=0$ is regular because $P(0)$ is not equal to 0

4 pts. (b) Give the approximating Euler equation.

$x^2P(0)y'' + xQ(0)y' + R(0)y = 0$

$P(x) = x+3$

$Q(x) = x^2+2x+1$

$R(x) = x^2$

$x^2(3)y'' + x(1)y' + (0)y = 0$

$\boxed{\text{Euler equation} \Rightarrow 3x^2y'' + xy' = 0}$

3 pts. (c) Give the indicial equation.

Assume $y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2}$

$3r(r-1)x^r + rx^r = 0$

$3r(r-1) + r = 0$

$3r^2 - 3r + r = 0$

$3r^2 - 2r = 0 \Rightarrow \boxed{r(3r-2) = 0}$

3 pts. (d) Use Theorem 5.6.1 on p. 293 of the text to describe the expected form of the solutions. **Do not find the coefficients of the series expansions!**

from indicial equation $r_1 = 0$ $r_2 = \frac{2}{3}$ $r_2 > r_1$ so use r_2

$\boxed{y_1 = x^{\frac{2}{3}} \sum_{n=0}^{\infty} a_n x^n}$ ✓

where a_n is generated using r_2

10∂

(10) A certain differential equation with a regular singularity at $x = 0$ has indicial equation $r(2r-1) = 0$. We substituted $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ into the differential equation and simplified, obtaining

$$(1) \qquad \sum_{n=0}^{\infty} (n+r)(2(n+r)-1)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0.$$

(a) Find a_1, a_2, a_3, and a_4 for the solution corresponding to the root $r = 1/2$, given that $a_0 = 1$. 5 pts.

$$\sum_{n=0}^{\infty} (n+r)(2n+2r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$k = n+2$, $n = k-2$ $\Rightarrow \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$

$$r(2r-1)a_0x^r + (1+r)(2+2r-1)a_1x^{1+r} + \sum_{n=2}^{\infty}\left[(n+r)(2n+2r-1)a_n + a_{n-2}\right]x^{n+r} = 0$$

$r = \frac{1}{2} \Rightarrow \frac{1}{2}\cancel{(1-1)}^{0}a_0x^{\frac{1}{2}} + (1+\frac{1}{2})(1+1)a_1x^{3/2}$

$a_1 = 0$

$(n+r)(2n+2r-1)a_n + a_{n-2} = 0 \quad n \geq 2$

$$a_n = \frac{-a_{n-2}}{(n+r)(2n+2r-1)}$$

$2n+1-1$

$r = \frac{1}{2} \Rightarrow a_n = \frac{-a_{n-2}}{(n+\frac{1}{2})(2n)} \quad n \geq 2$

$n=2 \quad a_2 = \frac{-a_0}{(2+\frac{1}{2})(4)} = \frac{-1}{\frac{5}{\cancel{2}} \cdot \cancel{4}2} = \frac{-1}{10}$

$n=3 \quad a_3 = \frac{-a_1}{(3+\frac{1}{2})(6)} = \frac{-0}{(3+\frac{1}{2})6} = 0$

$n=4 \quad a_4 = \frac{-a_2}{(4+\frac{1}{2})(8)} = \frac{+1}{10}\left(\frac{1}{\frac{9}{2}\cdot 8}\right) = \frac{1}{360}$

$$\boxed{\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{-1}{10} \\ a_3 &= 0 \\ a_4 &= \frac{1}{360} \end{aligned}}$$

(b) Use formula 1 to prove that for $r = 1$ (which is not a root of the indicial equation) $a_0 = a_1 = a_2 = a_3 = 0$. 4 pts.

$r=1 \qquad 1(2-1)a_0x^1 + (1+1)(2+2-1)a_1x^2 + \sum_{n=2}^{\infty}\left[(n+1)(2n+2-1)a_n + a_{n-2}\right]x^{n+1} = 0$

$\therefore a_0 = 0 \qquad \therefore a_1 = 0$

$$a_n = \frac{-a_{n-2}}{(n+1)(2n+1)}$$

$a_2 = \frac{-a_0}{1(4+1)} = 0$

$a_3 = \frac{-a_1}{4(6+1)} = 0$

$\therefore a_0 = a_1 = a_2 = a_3 = 0$

All coefficients of x^n must be equal to zero. This makes $a_0 = a_1 = 0$ which in turns makes all other $a_n = 0$ because of the relationship between a_n and a_{n-2}.

100

5 pts. (11) You are given that $y(x) = x^3$ is a solution of the following differential equation. Use the method of reduction of order to find a second independent solution. **Other methods will not receive credit!**

$$2x^2y'' - 5xy' + 3y = 0.$$

$y_1 = x^3 \qquad y_2 = u y_1$

$u' = y_1^{-2} e^{-\int p(x)dx}$

$p(x) = \frac{-5x}{2x^2} = \frac{-5}{2x}$

$\int p(x)dx = \int \frac{-5}{2x}dx = -\frac{5}{2}\ln x$

$-\frac{12}{2} + \frac{5}{2} = \frac{-7}{2}$

$\frac{d}{dx} x^{\frac{-5}{2}} = -\frac{5}{2}x^{\frac{-7}{2}}$

$u' = (x^3)^{-2} e^{-(-\frac{5}{2}\ln x)} = x^{-6} e^{\ln x^{5/2}} = x^{-6}x^{5/2} = x^{\frac{-7}{2}}$

$u = \int x^{\frac{-7}{2}}dx = -\frac{2}{5}x^{-\frac{5}{2}}$

$y_2 = -\frac{2}{5}x^{\frac{-5}{2}}x^3 = -\frac{2}{5}x^{\frac{-5}{2}+\frac{6}{2}} = -\frac{2}{5}x^{\frac{1}{2}}$

$\boxed{y_2 = -\frac{2}{5}x^{\frac{1}{2}}}$ ✓