The Exponential Series

1 Section 1

We consider the initial value problem

$$X' = AX \qquad X(0) = [1,1]^t \tag{1}$$

where

$$A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

Then (as you can check) $\det(A - \lambda I) = \lambda^2$ so the only eigenvalue is $\lambda = 0$. The equation $AX_o = 0X_o$ is equivalent with the system

$$\begin{aligned} x_o + 2y_o &= 0 \\ -4x_o - 2y_o &= 0 \end{aligned}$$

The corresponding eigenspace is spanned by $[-2,1]^t$ and the straight line solution is

$$Y_1(t) = e^{0t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

To solve our initial value problem (1), we attempt to find a constant C such that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = CY_1(0) = \begin{bmatrix} -2C \\ C \end{bmatrix}$$

No such C exists! The problem is that it takes two linearly independent vectors to span $\mathbb{R}^2$. We cannot hope to solve for the general initial value using only $Y_1(0)$.

We will solve our initial value problem in another, quite cleaver, technique. To explain this, consider, for a moment, the following initial value problem.

$$y' = 2y, \qquad y(0) = y_o$$

This is a separable equation and the solution is

$$y(t) = y_o e^{2t}$$

Now, recall that e^x is given by the power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots$$

Hence,

$$\begin{aligned} y(t) &= y_o e^{2t} \\ &= y_o + 2t\, y_o + \frac{2^2 t^2}{2!}\, y_o + \frac{2^3 t^3}{3!}\, y_o + \cdots + \frac{2^n t^n}{n!}\, y_o + \ldots \\ &= y_o + t\, 2y_o + \frac{t^2}{2!}\, 2^2 y_o + \frac{t^3}{3!}\, 2^3 y_o + \cdots + \frac{t^n}{n!}\, 2^n y_o + \ldots \end{aligned}$$

Is it conceivable that the solution to our initial value problem (1) can be expressed in exactly the same manner? i.e.

$$\begin{aligned} X(t) &= e^{At} X_o \\ &= X_o + t\, AX_o + \frac{t^2}{2!}\, A^2 X_o + \frac{t^3}{3!}\, A^3 X_0 + \cdots + \frac{t^n}{n!}\, A^n X_o + \ldots \end{aligned} \tag{2}$$

(We interpret quantities such as $A^3 X_o$ as $A(A(AX_o))$.) To check this, we compute

$$\begin{aligned} AX_o &= \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} \\ A^2 X_o = A(AX_o) &= \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Thus, $A^n X_o = \mathbf{0}$ for all $n \geq 2$, so our reasoning suggests that

$$X(t) = X_o + tAX_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 + 3t \\ 1 - 6t \end{bmatrix}$$

As a check, we compute that

$$AX(t) = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 1 + 3t \\ 1 - 6t \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = X'(t)$$

It is also clear that $X(0) = [1, 1]^t$, so we indeed have solved our initial value problem.

In general, it turns out that the series represented by formula (2) converges for any $n \times n$ matrix A and $n \times 1$ column vector X_o. Granted this, and granted that we may differentiate this series term by term, we can show that this series always represents a solution to the initial value problem $X' = AX$, $X(0) = X_o$. Specifically, from formula (2), we compute that $X(0) = X_o$ and

$$\begin{aligned} X'(t) &= AX_o + 2\frac{t}{2!}\,A^2 X_o + 3\frac{t^2}{3!}\,A^3 X_0 + \cdots + n\frac{t^{n-1}}{n!}\,A^n X_o + \dots \\ &= A(X_o + t\,AX_o + \frac{t^2}{2!}\,A^2 X_o + \frac{t^3}{3!}\,A^3 X_0 + \cdots + \frac{t^{n-1}}{(n-1)!}\,A^{n-1} X_o + \dots \\ &= AX(t) \end{aligned}$$

Usually, this infinite series of vectors is difficult to explicitly sum. In our example, we were aided by the fact that the seemingly infinite series (2) actually turned out to be a finite series because $A(AX_o) = \mathbf{0}$. Was this due to our choice of initial data? Suppose instead we had set $X_o = [a, b]^t$. Then

$$AX_o = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 2 \\ -4 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{3}$$

Hence

$$A(AX_o) = aA \begin{bmatrix} 2 \\ -4 \end{bmatrix} + bA \begin{bmatrix} 1 \\ 2 \end{bmatrix} = a\mathbf{0} + b\mathbf{0} = \mathbf{0}$$

Thus, for the given system, the same technique would work for any initial data.

Formula (2) can also be used to explain the eigenvalue-eigenvector technique we used earlier. Consider, for example, the system $X' = AX$, $X(0) = X_o$ where

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad X_o = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

We will compute the sum of the infinite series in (3). To do so, we write X_o as a linear combination of the eigenvectors for A. Specifically, it is easily computed that the eigenvalues are 4 and 2 and that the corresponding eigenvectors are $X_1 = [1,1]^t$ and $X_2 = [1,-1]^t$. Thus, $AX_1 = 4X_1$ and $AX_2 = 2X_2$. Also

$$X_o = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = X_1 - 3X_2$$

Then

$$\begin{aligned} AX_o &= AX_1 - 3AX_3 = 2X_1 - 3(4X_2) \\ A^2X_o &= A(AX_o) = A(2X_1 - 3(4X_2)) = 2AX_1 - 3(4AX_2) \\ &= 2^2X_1 - 3(4^2X_2) \end{aligned}$$

In general

$$A^nX_o = 2^nX_1 - 3(4^nX_2)$$

Hence

$$\frac{t^n}{n!}A^nX_o = \frac{2^nt^n}{n!}X_1 - 3\frac{4^nt^n}{n!}X_2$$

Summing these terms, we see that

$$X(t) = e^{2t}X_1 - 3e^{4t}X_2$$

which is the answer that we would have obtained using our earlier methods.

Exercises

1. Use the technique of the first example in this section to solve the $X' = AX$, $X(0) = X_o$ where

   (a)

   $$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad X_o = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   Ans:

   $$e^{tA} = \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix} \quad X(t) = \begin{bmatrix} 2+t \\ 1+t \end{bmatrix}$$

   (b)

   $$A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}, \quad X_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   Ans:

   $$e^{tA} = \begin{bmatrix} 1+3t & -t \\ 9t & 1-3t \end{bmatrix} \quad X(t) = \begin{bmatrix} 1+2t \\ 6t+1 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix}, \quad X_o = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$e^{tA} = \begin{bmatrix} 1-t & -t^2+t & t^2 \\ -t & 1-t-t^2 & t^2+2\,t \\ -t & -t-t^2 & 1+t^2+2\,t \end{bmatrix} \quad X(t) = \begin{bmatrix} 1+t+t^2 \\ 3\,t+2+t^2 \\ 3\,t+t^2+3 \end{bmatrix}$$

2. Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad X_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$e^{tA} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \quad X(t) = \begin{bmatrix} \cos(t) + 2\,\sin(t) \\ -\sin(t) + 2\,\cos(t) \end{bmatrix}$$

(a) Compute e^{tA} using formula (5) on page 6.

(b) Use your answer to solve the system $X' = AX$, $X(0) = X_o$ where X_o is as stated above.

Hint: You should discover that A^n has a different form, depending on whether n is odd or even. You should also discover that the sum of the odd terms is related to the series expansion for $\sin t$. What is the sum of the even terms related to?

2 Section 2

In the last section we showed that if A is an $n \times n$ matrix and X_o is an $n \times 1$ column vector, then the solution to the initial value problem $X' = AX$, $X(0) = X_o$ is

$$\begin{aligned} X(t) &= e^{At} X_o \\ &= X_o + t\,AX_o + \frac{t^2}{2!}\,A^2 X_o + \frac{t^3}{3!}\,A^3 X_0 + \cdots + \frac{t^n}{n!}\,A^n X_o + \ldots \end{aligned} \tag{4}$$

where we interpret quantities such as $A^3 X_o$ as $A(A(AX_o))$.

This can be conveniently expressed using the product of $n \times n$ matrices. In general, if A and B are matrices, we define AB to be the matrix each of whose columns is A times the corresponding column of B.

Example 1: Compute AB where A and B are as follows.

1.
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

2.
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & 3 \\ 6 & -5 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 7 & 1 \end{bmatrix}$$

Solution: By definition
$$AB = \begin{bmatrix} A\begin{bmatrix} 1 \\ 4 \end{bmatrix} & A\begin{bmatrix} 2 \\ 5 \end{bmatrix} & A\begin{bmatrix} 3 \\ 6 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 9 & 12 & 15 \\ 14 & 19 & 24 \end{bmatrix}$$
Similarly, for part (b),
$$AB = \begin{bmatrix} A\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} & A\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 14 & -1 \\ 14 & 6 \\ 19 & 9 \end{bmatrix}$$

The reader should note that there are some matrices which cannot be multiplied. For example,
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$$
is undefined because the columns of the matrix on the right have lenght 3 and the matrix on the left can only multiply columns of length 2. In general, an $m \times n$ matrix can only multiply an $n \times q$ matrix. The result will be an $m \times q$ matrix.

The significance of matrix multiplication for us is that it may be used to describe repeated multiplication of a vector times a matrix. Specifically, in linear algebra, it is shown that matrix multiplication is asoiciative–i.e. for any matrices A, B and C for which the product $A(BC)$ is defined
$$A(BC) = (AB)C$$
Thus, products such as $A(A(AX))$ may be expressed as $(A^3)X$ where $A^3 = A \cdot A \cdot A$. In particular, formula (1) may be expressed as X_o times the matrix
$$e^{At} = I + t\,A + \frac{t^2}{2!}\,A^2 + \frac{t^3}{3!}\,A^3 + \cdots + \frac{t^n}{n!}\,A^n + \dots \tag{5}$$

From this point of view, the calculation we did in the last section in solving the system

$$X' = AX, \qquad X(0) = [1, 1]^t \tag{6}$$

where

$$A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

can be explained as follows. Note that

$$A^2 = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence

$$e^{tA} = I + tA = \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix}$$

Hence, the solution to our initial value problem is

$$X(t) = e^{tA} X_o = \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+3t \\ 1-6t \end{bmatrix}$$

Notice what a wonderful thing e^{tA} is: the product of e^{tA} with X_o produces the solution with initial value X_o–there is no necessity to solve a system or the unknown constants.

We can use this example to make another important observation. Suppose that we want to solve the initial value problem

$$X' = AX, \quad X(0) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

where A is as above. Then from the preceding discussion, the answer is

$$\begin{aligned} X(t) &= \begin{bmatrix} 1+2t & t \\ -4t & 1-2t \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\ &= C_1 \begin{bmatrix} 1+2t \\ -4t \end{bmatrix} + C_2 \begin{bmatrix} t \\ 1-2t \end{bmatrix} \\ &= C_1 X_1(t) + C_2 X_2(t) \end{aligned}$$

where $X_1(t)$ and $X_2(t)$ are, respectively the first and second columns of $e^t A$. Thus *the columns of $e^t A$ form a fundamental set of solutions for the system.* The following theorem states that this is always true. The proof is identical with the computations from the preceding example.

Theorem 1. *Let A be an $n \times n$ matrix. Then the columns of e^{tA} form a fundamental set of solutions for the system $X' = AX$.*

The computation just done worked out due to the fact that for the particular matrix A in question, $A^2 = \mathbf{0}$. In general, an $n \times n$ matrix is said to be **nilpotent** if some power of A is zero. Nilpotent matrices are, of course, very special. Remarkably, however, they are more common than one might think, due to the following theorem from linear algebra, which is a special case of a more general result known as the Caley-Hamilton theorem.

Theorem 2. *Let A be an $n \times n$ matrix which has only one eigenvalue λ. Then $A - \lambda I$ is a nilpotent matrix.*

It turns out that this theorem tells us all that we need to solve the general 2 dimensonal system.

Example 1. Solve the initial value problem

$$X' = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} X \qquad X(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

Hence, A has $\lambda = 3$ as its only eigenvalue. According to Theorem 1, the following matrix B is nilpotent

$$B = A - 3I = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$$

Indeed, the reader can check that $B^2 = \mathbf{0}$. But then

$$e^{tA} = e^{t(B+3I)} = e^{tB+3tI}$$

If we were dealing with numbers, we could simplify this expression as

$$e^{3tI}e^{tB} \tag{7}$$

This simplification is still valid in our context, but its justification will require more thought than one might expect.

Posponing the justification for the moment, we may finish our example as follows.

$$e^{3tI} = I + 3t\,I + \frac{(3t)^2}{2!}\,I^2 + \cdots + \frac{(3t)^n}{n!}\,I^n + \cdots = e^{3t}I$$

Hence (since $B^2 = 0$)

$$e^{tA} = e^{3tI}e^{tB} = e^{3t}(I + tB) = e^{3t}\begin{bmatrix} 1-2t & 2t \\ -2t & 1+2t \end{bmatrix}$$

Therefore

$$X(t) = e^{3t}\begin{bmatrix} 1-2t & t2 \\ -2t & 1+2t \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = e^{3t}\begin{bmatrix} 2+2\,t \\ 2\,t+3 \end{bmatrix}$$

In order to finish the discussion of Example 1, we should justify the simplification (6). At first glance this seems to be a consequence of "laws of the exponent." Specifically, it seems to be a consequence of

$$e^{A+B} = e^A e^B$$

Unfortunately, as you will see in the exercises, this "law" *fails* for some matrices. It turns out that this law is valid only if A and B commute-i.e. satisfy $AB = BA$. However, for our purposes, all we really need to know is that

$$X(t) = e^{3t}e^{tB}X_o$$

satisfies $X' = AX$ and $X(0) = X_o$. The latter condition is clear and the former follows from the product formula for differentiation:

$$\begin{aligned}(e^{3t}e^{tB}X_o)' &= 3e^{3t}e^{tB}X_o + e^{3t}Be^{tB}X_o \\ &= (3I + B)e^{3t}e^{tB}X_o = AX\end{aligned}$$

It should be noted that Theorem 2, along with the method of eigenvalues and eigenvectors, allows us to solve any system two dimensional system $X' = AX$. If A has two different eigenvalues, then the eigenvalue-eigenvector method will yield the solution. If A has only one eigenvalue λ, then $B = A - \lambda I$ will be nilpotent and

$$e^{tA} = e^{t(B+\lambda I)} = e^{t\lambda}e^{tB}$$

Actually, to compute $e^{tA}X$, is not really necessary that there be an n such that $(A-\lambda I)^n = 0$; all we need is that there be an n such that $(A-\lambda I)^n X = 0$. Such X are referred to as generalized eigenvectors:

Definition 1. *Let A be an $n \times n$ matrix. An $n \times 1$ column vector X is a generalized eigenvector corresponding to λ if (1) $X \neq 0$ and (2) there is an n such that $(A - \lambda I)^n X = 0$. The smallest such n is called the "order" of X.*

Notice that a generalized eigenvector of order 1 is just an ordinary eigenvector. If X has order $n > 1$, then $Y = (A - \lambda I)^{n-1} X$ in non-zero and satisfies $(A - \lambda I)Y = 0$. Hence, Y is an ordinary eigenvector. In particular, λ must be an eigenvalue.

It turns out that we can solve any $n \times n$ system of linear differential equations by finding the generalized eigenvectors. This is due to the following theorem:

Theorem 3. *Suppose that A is an $n \times n$ matrix and λ_o is a root of order r of the characteristic polynomial $p(\lambda)$ of A. Then A has r linearly independent generalized eigenvectors corresponding to $\lambda = \lambda_o$. Furthermore, every generalized eigenvector X corresponding to the eigenvalue λ_o will satisfy*

$$(A - \lambda_o I)^r X = 0.$$

The following example illustrates how to use this theorem in the 3×3 case

Example 2. Find a fundamental set of solutions to the system $X' = AX$ where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution: We compute that

$$\det(A - \lambda I) = (3 - \lambda)^2 (5 - \lambda).$$

Hence the eigenvalues are 3 and 5.

It turns out that each eigenvalue yields only one independent eigenvector which are, respectively

$$\lambda = 3, \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \lambda = 5, \quad Y_1 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}. \tag{8}$$

These two vectors yield the solutions

$$x_1(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; y_1(t) = e^{5t} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} .$$

We need a third independent solution, which will come from the $\lambda = 3$ root since this root has multiplicity 2. To find it we use generalized eigenvectors. According to Theorem 3, the generalized eigenvectors X corresponding to $\lambda = 3$ satisfy

$$(A - 3I)^2 X = 0.$$

Let

$$B = A - 3I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \tag{9}$$

We compute

$$B^2 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

If $X = [x, y, z]^t$, the equation $B^2 X = 0$ is equivalent with $z = 0$. Hence, x and y are arbitrary. We set $x = s$ and $y = t$. Then

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \\ &= s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Our basis for the generalized eigenspace is

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} .$$

Using the observations that $A = B + 3I$ and $B^2 X_1 = 0$ we get the solution

$$\begin{aligned} x_1(t) &= e^{tA} X_1 \\ &= e^{3tI} e^{tB} X_1 \\ &= e^{3t} \left(I + tB + \frac{t^2 B^2}{2} + \dots \right) X_1 \\ &= e^{3t} \left(X_1 + tBX_1 \right) \end{aligned}$$

From (9) on page 11, $BX_1 = 0$ so

$$x_1(t) = e^{3t} X_1 = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

which is the same solution we found before. Similarly, we obtain a second solution

$$\begin{aligned} x_2(t) &= e^{3t} \left(X_2 + tBX_2 \right) \\ &= e^{3t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + te^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= e^{3t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Finally, our fundamental set of solutions is $\{x_1(t), x_2(t), y_1(t)\}$.

The technique illustrated in the preceding example is sufficient to any $n \times n$ system as long as we can find the eigenvalues. Specifically, if λ_1 is an eigenvalue of order r we:

1. Compute $B = A - \lambda_1 I$.
2. Compute $C = B^r$.
3. Solve the system $CX = 0$, expressing the general solution in the form

   $$X = s_1 X_1 + s_2 X_2 + \cdots + s_r X_r.$$

   The set $\{X_1, \dots, X_r\}$ is a basis for the corresponding generalized eigen space.

4. For each i we compute

$$\begin{aligned} x_i^1(t) &= e^{tA} X_i = e^{\lambda_1 t} e^{tB} X_i \\ &= e^{\lambda_1 t} \left(I + t\,B + \frac{t^2}{2!}\,B^2 + \cdots + \frac{t^r}{r!}\,B^r \right) X_i \end{aligned} \tag{10}$$

5. We do this process for each of the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_k$ obtaining solutions $x_j^i(t)$. The set of all solutions so obtained is a fundamental set of solutions for the system $X' = AX$.

Remark: We should prove the convergence of the series in formula (5). However, note that the series in (10) is finite so convergence is not an issue. We use this as justification for avoiding the convergence proof which does involve a substantial digression. However, we should at least explain what convergence means in this context. For each k,

$$E_k = I + t\,A + \frac{t^2}{2!}\,A^2 + \frac{t^3}{3!}\,A^3 + \cdots + \frac{t^k}{k!}\,A^k$$

is an $n \times n$ matrix. Let $E_k(i,j)$ denote the ij entry of E_k. Saying that the series in formula 2 converges is equivalent with saying that for all i and j, $\lim_{k\to\infty} E_k(i,j)$ converges. If $E(i,j)$ denotes the limit, then we say that the matrix $E = [E(i,j)]$ is the sum of the series. The main theorem concerning the convergence is

Theorem 4. *The series in formula (5) converges for all $n \times n$ matrices A and all t*

Exercises

1. For the matrices A, B and C, demonstrate the associative law $A(BC) = (AB)C$ by directly computing the given products in the given orders.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \end{bmatrix}$$

2. In Exercise 1, is $(CA)B$ defined? $B(AC)$? $A(CB)$?

3. Find a pair of 2×2 matrices A and B of your own choice such that $AB \neq BA$. [*Hint*: This isn't hard. It is a theorem that with probability 1, any two randomly selected matrices will not commute.]

4. Find all 2×2 matrices B such that $AB = BA$ where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

5. Find a pair of 2×2 matrices A and B such that $(A+B)(A+B) \neq A^2 + 2AB + B^2$. Under what conditions does this equality hold?

6. Find a pair of 2×2 matrices A and B such that $(A+B)(A-B) \neq A^2 - B^2$. Under what conditions does this equality hold?

7. Use the technique of Example 1 on page 8 to solve the initial value problem $X' = AX$, $X(0) = X_o$ where

(a)

$$A = \begin{bmatrix} 1.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} \qquad X_o = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Ans:

$$e^{tA} = \left[\begin{array}{cc} 1/2\, \mathrm{e}^t\,(2+t) & -1/2\, t\mathrm{e}^t \\ 1/2\, t\mathrm{e}^t & -1/2\, \mathrm{e}^t\,(-2+t) \end{array} \right] X(t) = \left[\begin{array}{c} 1/2\, \mathrm{e}^t\,(2+t) - t\mathrm{e}^t \\ 1/2\, t\mathrm{e}^t - \mathrm{e}^t\,(-2+t) \end{array} \right]$$

(b)

$$A = \begin{bmatrix} -11 & 27 \\ -3 & 7 \end{bmatrix} \qquad X_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ans:

$$e^{tA} = \left[\begin{array}{cc} -\mathrm{e}^{-2\,t}\,(-1+9\,t) & 27\, t\mathrm{e}^{-2\,t} \\ -3\, t\mathrm{e}^{-2\,t} & \mathrm{e}^{-2\,t}\,(1+9\,t) \end{array} \right] \quad X(t) = \left[\begin{array}{c} -\mathrm{e}^{-2\,t}\,(-1+9\,t) + 27\, t\mathrm{e}^{-2\,t} \\ -3\, t\mathrm{e}^{-2\,t} + \mathrm{e}^{-2\,t}\,(1+9\,t) \end{array} \right]$$

(c) Solve the initial value problem $X' = AX$, $X(0) = X_o$ where

$$A = \left[\begin{array}{cc} -4 & 3 \\ -12 & 8 \end{array} \right] \qquad X_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Ans:

$$e^{tA} = \left[\begin{array}{cc} -\mathrm{e}^{2\,t}\,(-1+6\,t) & 3\, t\mathrm{e}^{2\,t} \\ -12\, t\mathrm{e}^{2\,t} & \mathrm{e}^{2\,t}\,(1+6\,t) \end{array} \right] \quad X(t) = \left[\begin{array}{c} -\mathrm{e}^{2\,t}\,(-1+6\,t) + 3\, t\mathrm{e}^{2\,t} \\ -12\, t\mathrm{e}^{2\,t} + \mathrm{e}^{2\,t}\,(1+6\,t) \end{array} \right]$$

8.
$$A = \begin{bmatrix} -7 & 3 \\ -18 & 8 \end{bmatrix} \qquad X_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ans:

$$e^{tA} = \begin{bmatrix} 3\,e^{-t} - 2\,e^{2\,t} & e^{2\,t} - e^{-t} \\ -6\,e^{2\,t} + 6\,e^{-t} & -2\,e^{-t} + 3\,e^{2\,t} \end{bmatrix} X(t) = \begin{bmatrix} 2\,e^{-t} - e^{2\,t} \\ -3\,e^{2\,t} + 4\,e^{-t} \end{bmatrix}$$

9. Let A, B be as shown. Compute e^{tA}, e^{tB} and $e^{t(A+B)}$ [*Hint:* Let $C = A + B$ and use the series 2. Your final answer will involve sines and cosines]. Use your answer to show that $e^{t(A+B)} \neq e^{tA}e^{tB}$. Finally, show that $AB \neq BA$.
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

10. Solve the initial value problem $X' = AX$, $X(0) = [4, 2, 2]^t$ Note: To save time, we have provided two eigenvectors X_1 and X_2 for A. Furthermore, the eigenvalue for X_1 has multiplicity 2.

(a)
$$A = \begin{bmatrix} 0 & 19 & -10 \\ -3 & 16 & -4 \\ -3 & 7 & 5 \end{bmatrix} \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 17 & -10 & -1 \\ 4 & 4 & -2 \\ 3 & -6 & 9 \end{bmatrix} \quad X_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(c)
$$A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & -1 & 0 \\ -2 & -2 & 4 \end{bmatrix} \quad X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} X_2 = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

11. Find e^{tA} for the 2×2 matrices in Exercises 4 and 10 on p. 428. Then use your answer to find the general solution to the system.

12. Find e^{tA} for the following matrix. Then use your answer to find the general solution to the system $X' = AX$.

$$A = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Ans:

$$e^{tA} = \begin{bmatrix} 1 & 2t & t^2 + t & -1/3\,t^3 - 1/2\,t^2 \\ 0 & 1 & t & -1/2\,t^2 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $x(t) = c_1x_1(t) + c_2x_2(t) + c_3x_3(t) + c_4x_4(t)$ where

$$x_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad x_2(t) = \begin{bmatrix} 2t \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad x_3(t) = \begin{bmatrix} t^2 + t \\ t \\ 1 \\ 0 \end{bmatrix} \quad x_4(t) = \begin{bmatrix} -1/3\,t^3 - 1/2\,t^2 \\ -1/2\,t^2 \\ -t \\ 1 \end{bmatrix}$$

13. Find e^{tA} for the following matrix. Then use your answer to find the general solution to the system $X' = AX$.

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Ans: The answers are just the answers to Exercise 12 multiplied by e^{3t}.

14. Let A be the 3×3 matrix in Exercise 11 on p. 436 of the text.

    (a) Find a basis $\{X_1, X_2\}$ for the space of generalized eigenvectors corresponding to the eigenvalue $r = 1$.

    (b) Find $e^{tA}X_1$ and $e^{tA}X_2$.

    (c) Use the answer from 14b to find the general solution to the given system, given that an eignevector corresponding to $r = 2$ is

$$Y_o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

15. A certain 6×6 matrix A has characteristic polynomial $(r-2)^4(r-3)^2$. Let X be a generalized eigenvector for A corresponding to $r = 2$. Give a formula for $e^{tA}X$ that does not require summing an infinite series. Your formula should use as few matrix products as possible relative to the given information.

16. Repeat Exercise 15 for an generalized eigenvector Y corresponding to $r = 3$.

17. A certain 5×5 matrix A has characteristic polynomial $-(r-1)^5$. Give a formula for $e^{tA}X$ that does not require summing an infinite series. Your formula should use as few matrix products as possible relative to the given information.

18. A certain real 3×3 matrix A has eigenvalues 3 and $-2-i$ with corresponding eigenvectors

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, X_2 \begin{bmatrix} 1 \\ 2 \\ 2+i \end{bmatrix}.$$

Give a fundamental set of **real** solutions for the equation $X'(t) = AX(t)$.