Interest Theory

Richard C. Penney
Purdue University

Contents

CHAPTER 1

Compound Interest

1. The TI BA II Plus Calculator

A financial calculator approved by the SOA is required for the Course 2 Actuarial Exam. *We recommend the TI BA II Plus, either the solar or battery version.* We will discuss some of the details of its operation in these notes as the need arises. However, a few initial comments are in order.

You will probably want to change some of the default settings for the BAII. Changing the defaults is accomplished by pressing [2nd][FORMAT] and using the up/down arrows to browse the format menu. Changes are made either by entering a value or pressing [2nd][SET] to change a choice. *The changes you make remain permanent until you either replace the battery or enter* [2nd][RESET]. To exit the FORMAT menu, press [CE/C].

Some changes you might want to make are:

1. By default, the BA II rounds the display to 2 decimal accuracy, despite the fact that it stores them, and works with them, to a much higher degree of accuracy. This can present a problem if you, say, record answers on paper for subsequent use in further calculations; two decimal accuracy is not sufficient for many applications. We suggest that you reset the value of the **DEC** variable in the FORMAT menu from 2 to 8.
2. By default, the BA II does not follow the standard rules concerning the order of operations. Instead, it computes numbers in the order they are entered. Thus, entering $2+3*6$ produces $5*6 = 30$ rather than $2+18 = 20$. To get the BA II to behave more like a standard scientific calculator, we suggest that you reset the **Cnn** (chain calculation) variable to **AOS**(Algebraic Operation System) by scrolling down to Cnn and pressing [2nd][SET].
3. If you use the BA II for other classes, you might also want to change the **DEG** variable to **RAD**.

The BA II remembers the values of all registers, even when turned off. Hence, *every time you begin a new session*, you should enter [2nd][CLR WORK] and [2nd][CLEAR TVM] to clear out all registers. Don't use [2nd][RESET] as this will also reset the defaults. Similarly, you should enter [2nd][CLEAR TVM] at the beginning of each new problem if you have accessed TVM features of the BA II. If you have used other work sheets, you should also enter [2nd][CLR WORK].

Remark. Financial calculators are not as useful on the Course 2 Actuarial Exam as one might expect. Few of the problems may be solved using only the calculator. *It is essential that you learn to use the formulas as well as how to use the calculator.* For this reason, we will insist, at least initially, that you write a formula for the solution to all assigned homework problems, even if you do the computations on the calculator.

2. Compound Interest

The simplest example of interest is a loan agreement two children might make: "I will lend you a dollar, but every day you keep it, you owe me one more penny." In this example, the interest rate is 1%/day and the amount owed after t days is

$$A(t) = 1 + .01t$$

In this formula, the quantity $.01t$ is the interest at time t. (In general, the interest is the difference between what was borrowed and what is owed.)

REMARK. In the above example, we can describe the interest rate as a percent (1%) or as a numeric value (.01). *When we state an interest rate we will always mean a numeric value, and not a percent, unless we indicate otherwise.*

In these notes, we use the year and the dollar as our fundamental units as this is most common in actuarial science. We will assume that, unless otherwise stated, all interest rates are per unit time–i.e. per year. However, the reader should be aware that all of our formulas are valid regardless, of the units of measure.

If, as above, the interest is proportional to time, then we say that the interest is *simple interest.*

DEFINITION 1. *An quantity grows at a rate i simple interest if the amount at time t is given by*

$$A(t) = (1 + it)P \tag{1}$$

for some constant P. $A(t)$ is also referred to as the "future value at time t of P at simple interest rate i." P is referred to as the "present value at time 0 of the account at simple interest rate i."

EXAMPLE 1. On Jan. 1 of a non-leap year, I invest \$5,000 at 3% simple interest. How much do I have on May 1? How much would I have in 3 years?

SOLUTION. On May 1, I have had the money for $3 \cdot 31 + 30 = 123$ days, which is 123/365th of a year. Hence, I have

$$(1 + \frac{123}{365}.03)5000 = 5050.55$$

dollars.

In 3 years, I have

$$(1 + 3(.03))5000 = 5450.00$$

REMARK. In computing interest, it is typically assumed that interest is earned only on either the first day the account is open or the last day, but not on both. Which day doesn't matter in computing the interest. Thus, in Example 1, it is correct not to count the interest earned on May 1.

The question of how many days are in a year is actually somewhat complicated. The most obvious answer is that a year will have either 365 or 366 days, depending on whether or not it is a leap year. It has to be remembered, however, that accounting practices became standardized long before even hand held calculators were available, not to mention personal computers. Thus, many schemes have been developed to simplify hand computations.

For example, it is common to not give interest on Feb. 29, in which case all years effectively have 365 days. Another method, referred to as *exact interest,* is to give interest on leap day, but still say that all years have 365 days. Thus, under this standard, at the nth day of the year, P dollars invested at rate i simple interest grows to

$$(1 + \frac{n}{365} i)P$$

In particular, at the end of a leap year, you have

$$(1 + \frac{366}{365} i)P$$

dollars.

There is another method, *ordinary interest,* (not to be confused with "simple interest") in which it is assumed that all months have 30 days and every year has 360 days! Thus, if you opened an 4% account on Jan. 1 1950 and closed it on May 10, 2002, you held your money for 51 years, 4 months and 10 days which, according to the rules of ordinary interest, is

$$51 \cdot 360 + 4 \cdot 30 + 10 = 18490$$

days. Hence P dollars invested at rate i simple interest will have grown to

$$(1 + \frac{18490}{360} .04)P$$

dollars. Ordinary interest has the feature that each month is 1/12 of a year. *In this class, unless otherwise stated, we will use ordinary interest as this greatly simplifies counting days.*

There is also something called *Banker's rule,* in which every year has 360 days, but you count the exact number of days you have held the money in computing the interest. To use Banker's Rule on the preceding example, you would have to count the days between Jan. 1, 1950 and May 10, 2002 and use this number instead of the 18490.

The use of exact interest is common in Canada while the Banker's rule is common in the US and in international markets.

Compound interest is much more common than simple interest. Suppose, for example, that I invest P dollars at rate i, compounded yearly. As with simple interest, at the end of the year, I have

$$A(1) = (1 + i)P$$

dollars.

With compound interest, however, I earn interest on the total amount on deposit at the beginning of the compounding period, not just the original principal. Hence, in another year, my account will again grow by a factor if $(1 + i)$, yielding

$$(1 + i)^2 P$$

dollars. After n years, I have

$$A(n) = (1 + i)^n P \tag{2}$$

dollars.

EXAMPLE 2. At the end of 1980, I deposited \$1,000 in an account that earns 7.3% interest, compounded yearly. How much did I have at the end of 2000, assuming that no further deposits or withdrawals are made?

Solution. My funds were on account from Dec. 31, 1980 to Dec. 31, 2000: a full 20 years. Hence, I have

$$(1.073)^{20}1000 = 4,092.55$$

dollars.

What if, in Example 2, I were to close my account after having left my money on deposit for only 6 months; how much would I get? The answer depends on the rules of the bank. Some accounts charge a substantial penalty for early withdrawal, meaning that you could actually lose money. In some cases, the bank uses simple interest for partial periods, in which case you would get

$$(1 + \frac{.073}{2})1000 = 1,036.50$$

dollars since the money was on deposit for a half year. Finally, we might simply substitute $n = 1/2$ into formula (2) yielding

$$(1.073)^{1/2}1000 = 1,035.86$$

In practice, this last method is probably the least common. However, in the mathematical theory of interest, *if we say that an account earns compound interest at a rate i, we are implicitly stating that we use formula (2) for partial periods as well:*

Definition 2. *An quantity grows at a rate i compound interest if the amount at time t is given by*

$$A(t) = (1 + i)^t P \tag{3}$$

for some constant P. $A(t)$ is also referred to as the "future value at time t of P at compound interest rate i." P is referred to as the "present value at time 0 of the account at compound interest rate i."

In interest theory, we often use $A(t)$ (the amount function) to indicate the value of the account at time t. The function

$$a(t) = \frac{A(t)}{A(0)}$$

is referred to as the *accumulation function*. Thus, the accumulation function for compound interest is

$$a(t) = (1 + i)^t.$$

Example 3. Banks A and B both offer savings accounts that pay 5% interest per year. Bank A compounds yearly but uses simple interest for partial periods while bank B uses straight compound interest for all times. Compare the amount that you would have after 3 years and 2 months if you invested $2,000 in bank A with the same investment in bank B.

Solution. In bank A, at the end of 3 years, you have

$$(1.05)^3 2000 = 2315.25$$

dollars. For the next 2 months you earn simple interest on $2,315.25 dollars, yielding

$$(1 + .05(\frac{2}{12}))2315.25 = 2334.54$$

In bank B you have

$$(1.05)^{\frac{38}{12}}2000 = 2334.15$$

This example makes an important point: the difference between using simple interest for partial periods verses compound interest is slight. In fact, in Figure 2 we have graphed the amount of money in banks B and A on the same graph for $0 \leq t \leq 1$. The graphs are so close that they appear to be one single graph. Mathematically, we are saying thet for "small" values of i and t,

$$(1_i)^t \approx 1 + ti. \tag{4}$$

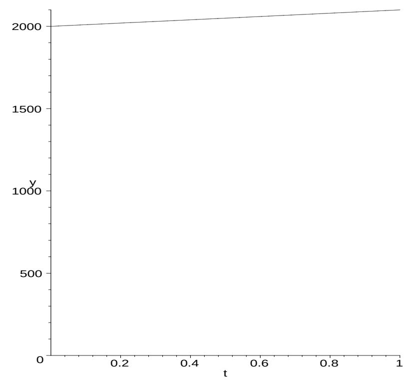

FIGURE 1. There are two graphs here!

Often banks offer accounts which compound at intervals other than one year. For example, a bank might offer an account that pays 6% interest, compounded four times a year. What this means is that every quarter of a year, the account grows by $\frac{6}{4}$%. Thus, in one year, P dollars grows to

$$(1 + \frac{.06}{4})^4 P = (1.0613)P$$

This is the same growth as an account at 6.13% interest, compounded annually. This 6.13% is called the *annual effective yield* while the "6%" interest rate is referred to as the *nominal* rate, in that it's the rate that the bank might name when describing the account.

In general, the symbol $i^{(n)}$ indicates an interest rate i which is compounded n times a year. Thus, the discussion in the preceding paragraph says that an interest rate of $.06^{(4)}$ is the same as $.0613^{(1)}$. The rate $i^{(n)}$ is equivalent with the annual effective rate j where

$$(1 + \frac{i^{(n)}}{n})^n = 1 + j$$

EXAMPLE 4. A bank offers an account that yields a nominal rate of return of 3.3% per year, compounded quarterly. What is the annual effective rate of return? How many years will it take for the balance to double?

SOLUTION. Since each year has 4 quarters, P dollars at the beginning of the year grows to

$$(1 + \frac{.033}{4})^4 P = (1.0334)P$$

by the end of the year. Hence, the annual effective rate of interest is 3.34%.

To compute how long it takes for the account to double, we can either work in quarters or years. In quarters, we seek n so that

$$(1 + \frac{.033}{4})^n P = 2P$$
$$(1 + \frac{.033}{4})^n = 2$$
$$n \ln(1.00825) = \ln 2$$
$$n = \frac{\ln 2}{\ln(1.00825)} = 84.36$$

The number of years is $84.36/4 = 21.09$.

Since our effective rate of return is 3.34% per year, we can find the answer directly in years as follows:

$$(1.0334)^n P = 2P$$
$$(1.0334)^n P = 2$$
$$n \ln(1.0334) = \ln 2$$
$$n = \frac{\ln 2}{\ln(1.0334)} = 21.1$$

The answer differs slightly from that found previously due to round off error. Specifically, 3.34% is only an approximation of the annual effective yield. A more exact value is 3.3410626%, which does yield the same answer as before.

Actually, both answers might be wrong. If the bank only credits interest each quarter, then the doubling would not occur until after the 85th quarter, in which case the correct answer is $21\frac{1}{4}$ years.

This problem may also be solved on the BA II. Explicitly, enter

0[PMT] (We are not making payments into the account)

[2nd][P/Y]4[ENTER] (four compounding periods per year)

[CE/C] (return to calculator mode)

3.3[I/Y] (3.3 % nominal interest rate per year)

1[PV] (present value is one–i.e. we deposit $1)

4[N] (number of compounding periods)

[CPT][FV] (compute the future value of the account)

yielding $FV = -1.0334$. The value is negative because the final balance is considered as a final withdrawal; hence it is negative. This tells us that an investment of $1 grows to $1.0334, making the annual effective yield .0334% as before. Instead of entering 4[N], we could have entered 1[2nd][xP/Y][N] since [2nd][xP/Y] converts the number of years into the number of payments.

To solve the second part of the problem, we enter

-2[FV]

[CPT][N]

getting $N = 84.36$ quarters. We can then divide by 4 to get the answer. Note that the rest of the data does not need to be reentered since it has not changed.

REMARK. The preceding example makes an important point: *If deposits are entered into the BA II as positive quantities, the [FV] key yields the* **negative** *of the actual future value.* Thus, one should enter [+/-]before using this value in other calculations. Similarly, *if the future value is entered as a positive quantity, then both the present value and the payments are the negative of their actual values.*

EXAMPLE 5. Bank A offers a nominal rate of 5.2% interest, compounded twice a year. Bank B offers 5.1% interest, compounded daily. Which is the better deal?

SOLUTION. We convert each nominal rate into an annual effective rate:

Bank A

$$(1 + \frac{.052}{2})^2 = 1.052676$$

for a 5.27% annual effective rate of return.

Bank B We recall that under ordinary interest, years have 360 days. Hence, the annual rate of return is

$$(1 + \frac{.051}{360})^{360} = 1.052319218$$

for a 5.23% annual effective rate of return. It's darn close, but Bank A wins.

REMARK. Daily compounding is very common. I recently called Huntington Bank in Lafayette to ask what the current interest rate on 5 year CD's was. I was told something like 5.2% with an annual effective yield of 5.23%. I asked how often it is compounded. The answer was daily. Daily compounding eliminates the problem of partial periods: you get whatever the balance was at the end of the preceding day.

EXAMPLE 6. On Jan.1, 1998, I open an account with a \$1000 deposit. On Jan.1, 1999, I withdraw \$500 and on Jan.1, 2001 I deposit \$1,500. If the account earns 7.5% interest, compounded yearly, and no further deposits or withdraws are made, what was the balance on Jan.1, 2002?

SOLUTION. It is often helpful to draw a "time line" (Figure 2) to indicate what deposits were made when:

There are two ways to solve this problem; easy and easier. First, the easy way:

The balance on Jan. 1, 1999 was one year interest on \$1000, minus \$500:

$$1000(1.075) - 500 = 575$$

The balance on Jan. 1, 2001 was 2 years interest on \$575, plus the \$1,500 deposit:

$$575(1.075)^2 + 1500 = 2164.48$$

My final balance is 2 years interest on \$2164.48:

$$2164.48(1.075)^2 = 2501.34$$

Now for the easier way. Without any further deposits, our \$1000 would have grown to

$$1000(1.075)^5 = 1435.63$$

Withdrawing \$500 caused us to loose both the \$500 as well as its interest for the next 4 years; a net loss of

$$500(1.075)^4 = 667.73$$

Finally, the \$1,500 deposit was on account for 2 years, yielding a total of

$$1500(1.075)^2 = 1733.44$$

Hence, our balance is

$$1435.63 - 667.73 + 1733.44 = 2501.34$$

as before.

In general, we may treat deposits and withdrawals separately: The balance $B(t)$ at time t in an account that earns compound interest at rate i is given by the formula

$$B(t) = B(0)(1+i)^t + C_1(1+i)^{t-t_1} + \cdots + C_n(1+i)^{t-t_n} \tag{5}$$

where $B(0)$ is the initial balance, C_i are all of the deposits/withdrawals made between time 0 and time t, (withdrawals are considered as negative deposits) and t_i is the time at which the deposit/withdrawal C_i was made. Equation (5), or any equation equivalent to it, is referred to as *the equation of value.*

EXAMPLE 7. Ed invests \$550 at 3% interest. At the end of the first year, he withdraws \$100, at the end of the second year, he withdraws \$300 and at the end of the third year he deposits an additional \$50 at the same interest rate. He closes the account at the end of the fourth year. What was his final withdrawal?

SOLUTION. The time line is indicated in Figure 2. We treat each deposit and withdrawal separately. The deposits, together with interest, total to

$$(1.04)^4 550 + (1.04)50 = 695.42$$

Each withdrawal reduces both the balance and the future interest. Thus, the withdrawals up to the end of year 4 reduce the balance by

$$(1.04)^3 100 + (1.04)^2 300 = 436.97$$

Thus, Ed still has

$$695.42 - 436.97 = 258.45$$

which is his final balance.

One of the most important concepts in interest theory is that of present and future value.

EXAMPLE 8. How much must I deposit today into an account that pays 6.4% to be able to pay you \$500, two years hence?

SOLUTION. Let the amount deposited be P. We need to solve the equation

$$(1.064)^2 P = 500$$
$$P = (1.064)^{-2} 500 = 441.66$$

dollars.

On the BA II Plus, we would enter 0[PMT], [2nd][P/Y]1, [CE/C], 6.4[I/Y], −500[FV], 2[N], [CPT][PV].

The preceding example makes an extremely important point: a promise to pay $500, two years from today is not worth $500 today: if we can invest money at 6.4%, $500 two years from now is only worth $441.66 today. We say that the *present value* of $500 two years from now at 6.4% interest is $441.66. Equivalently, at 6.4% interest, $441.66 will grow to $500. Hence, the *future value* of $441.66 two years from now at 6.4% interest is $500.

The following definition is just a restatement of Definition (2) in different terms. These concepts are, however, sufficiently important to bear repeating.

DEFINITION 3. *The* future value *(FV) of P dollars at interest rate i, t years from now, is the amount that P dollars will grow to in n years. Hence*

$$FV = (1+i)^t P \tag{6}$$

The present value *(PV) of P dollars at interest rate i, t years from now, is the amount we would need to invest now to yield P dollars t years from now. Hence*

$$PV = (1+i)^{-t} P \tag{7}$$

The quantity $(1+i)^{-1}$ occurs so often that it has a special symbol:

$$(1+i)^{-1} = \nu$$

Hence, Formula 7 is often written

$$PV = \nu^t P$$

EXAMPLE 9. On Jan. 1, you won a "$400,000 sweepstakes." The prize is to be paid out in 4 yearly installments of $100,000 each with the first paid immediately. Assuming that you can invest funds at 5% interest compounded yearly, what is the present value of the prize?

SOLUTION. The time line is indicated in Figure 2.

If you invest each $100,000 payment at 5% interest, in 3 years you will have

$$(1.05)^3 100000 + (1.05)^2 100000 + (1.05)100000 + 100000 = 431012.50$$

The present value is

$$(1.05)^{-3} 431012.50 = 372324.80$$

dollars. Note that the award is actually worth considerably less than the advertised $400,000.

REMARK. We could have done this calculation in one step:

$$\begin{aligned}
&(1.05)^{-3}((1.05)^3 100000 + (1.05)^2 100000 + (1.05)100000 + 100000) \\
&= 100000 + (1.05)^{-1} 100000 + (1.05)^{-2} 100000 + (1.05)^{-3} 100000 \\
&= 100000 + \nu 100000 + \nu^2 100000 + \nu^3 100000 = 372324.80
\end{aligned}$$

In the last equality we are just summing the present value of each payment.

It is a general principle that *the value today of a promised series of future payments is the sum of their present values, computed at the prevailing interest rate for comparable investments.*

The following example illustrates a common type of problem, both in finance and on actuarial exams.

Example 10. At what time would a single payment of \$400,000 be equivalent the series of payments in Example 9–i.e. at what time t does the present value of \$400,000 equal the present values of the payments in Example 9?

Solution. We present two solutions; an exact solution and an approximate solution. For the exact solution, we note that from the solution to Example 9, the present value of all of the payments in question is 372324.80. Hence, we seek t such that

$$400000(1.05)^{-t} = 372324.80$$

which is equivalent with

$$\begin{aligned}(1.05)^{-t} &= \frac{372324.80}{400000}\\ -t\ln 1.05 &= \ln(372324.80/400000)\\ t &= 1.469516586\end{aligned}$$

For the approximate method, we note first that equating present values yields

$$400000(1.05)^{-t} = 100000 + (1.05)^{-1}100000 + (1.05)^{-2}100000 + (1.05)^{-3}100000$$

We use the approximation (4) which, in the current context, states $(1.05)^{-t} \approx 1 - .05t$. We find

$$400000 - .05t(400000) \approx 400000 - .05(100000) - .05(200000) - .05(300000)$$

yielding

$$t = \frac{.05(100000 + 200000 + 300000)}{.05(400000)} = 1.5 \tag{8}$$

Example 10 illustrate a general problem type in which we are given a series of n payments with values $C_1, C_2, \ldots, C_n$ made at times $t_1, t_2, \ldots, t_n$ respectively. We are asked to find the time t at which a single payment of $C_1 + C_2 + \cdots + C_n$ is equivalent with the given sequence of payments at a given interest rate i. Such problems may always be solved exactly as in the first part of the solution to Example 10. They may also be solved approximately using the approximation (4). This approximate method is referred to as **the method of equated time** and is important. The approximate time value obtained using this method is often denoted "$\overline{t}$" One can write down a general formula for the equated time solution. However, we feel that it is better to simply apply approximation (4) to the equation of value. It is important to note that the ".05" factor cancelled in equation (8). This demonstrates a general principle: *the approximation $\overline{t}$ obtained by the method of equated time does not depend on the interest rate i*

Multiplying both sides of formula (5) by $(1+i)^{-t}$ yields

$$0 = B(0) + C_1(1+i)^{-t_1} + \cdots + C_n(1+i)^{-t_n} - B(t)(1+i)^{-t}$$

If we consider $B(0)$ as an initial deposit C_0 and $-B(t)$ as a final withdrawal C_{n+1}, this formula becomes

$$0 = C_0 + C_1(1+i)^{-t_1} + \cdots + C_n(1+i)^{-t_n} + C_{n+1}(1+i)^{-t} \tag{9}$$

Each of the terms in the sum on the right is, then, the present value at time 0 of either a deposit/withdrawal. Specifically this formula says that *the sum of the present values of all deposits/withdrawals is zero, where the initial balance is considered as an initial deposit and the final balance as a final withdrawal.* Equation 9 is referred to as *the equation of value.* Solving for C_0 tells us that *the initial balance* *is the present value of the future withdrawals, minus the present value of the future deposits, where the final balance is considered as a final withdrawal.* Computing C_0 in this manner is referred to as the *prospective method* because it uses future activity in the account to compute the balance.

EXAMPLE 11. The balance on Jan. 1, 2003 in an account that earns compound interest at rate 3.2% per year was \$2,500. What was the balance on Jan. 1, 2000, given that the activity on the account was as described in the following chart.

Date	3/1/00	6/1/01	1/1/02	6/1/02
Dep./With.	300	−700	600	−200

SOLUTION. The time line is indicated in Figure 2.
According to formula (9), the initial balance is

$$\begin{aligned} C_0 &= -(1.032)^{-2/12}300 + (1.032)^{-17/12}700 \\ &- (1.032)^{-24/12}600 + (1.032)^{-29/12}200 + (1.032)^{-36/12}2500 \\ &= 2267.57 \end{aligned}$$

We can use formula (9) to compute the balance in the account at any point in time. We simply define this time to be time 0.

EXAMPLE 12. What was the balance in the account from Example 11 on 10/1/01?

SOLUTION. The \$600 was deposited 3 months after 10/1/01, the \$200 withdrawal occurred 8 months after 10/1/01, and the balance was \$2500, 15 months after 10/1/01. Hence, according to formula (9), the balance was

$$-(1.032)^{-3/12}600 + (1.032)^{-8/12}200 + (1.032)^{-15/12}2500 = 2004.02$$

Since, from the solution to Example 11, we know the initial balance, we can also use formula (5) to compute the balance:

$$(1.032)^{21/12}2267.57 + (1.032)^{19/12}300 - (1.032)^{4/12}700 = 2004.02$$

Computing the balance using formula (5) is referred to as the *retrospective method* because it uses past activity in the account to compute the balance. The retrospective method is useful when we know the initial balance while the prospective method is useful when we know the final balance.

Exercises

Calculate each of the following:

1. You invest \$500 at 6.4% simple interest per year.
   (a) How much is in your account after 1 year?
   (b) How much is in your account after 5 years?
   (c) How much is in your account after 1/2 year?
   (d) How much is in your account after 5 years and 6 months?
2. Redo Exercise 1 using *compound* interest instead of simple interest.
3. Redo Exercise 1 assuming that the account earns *compound* interest for integral time periods and simple interest for fractional time periods.
4. How long will it take for \$1,000 to accumulate to \$2000 at 5% annual compound interest?
5. Value of \$1250 invested for 4 years at 5% simple interest.

6. Value of \$1250 invested for 4 years at 5% compound interest .
7. Value of \$624 invested for 3 years at 6% simple interest.
8. Value of \$624 invested for 3 years at 6% compound interest .
9. Value of \$624 invested for 3 years at 6% compounded quarterly.
10. Value of \$3150 invested for 1 year at 4% simple interest.
11. Value of \$3150 invested for 1 year at 4% compound interest .
12. Value of \$8635 invested for 8 years at 5% compounded monthly.
13. Amount you need to invest now to have \$5000 in 4 years if your account pays 6% simple interest.
14. Amount you need to invest now to have \$5000 in 4 years if your account pays 6% compound interest .
15. Amount you need to invest now to have \$5000 in 4 years if your account pays 6% compounded monthly.
16. Amount you need to invest now to have \$100000 in 15 years if your account pays 5% compounded monthly.
17. Your account had \$486 in it on October 1, 1989 and \$743 in it on October 1, 1997. Assuming that no additions or withdrawals were made in the meantime, what annual effective interest rate accounts for the growth in the balance?
18. What is the annual effective interest rate that would account for a CD increasing in value from \$4000 on October 1, 1992 to a value of \$5431.66 on October 1, 1997?
19. One bank is paying 4.8% compounded monthly. Another bank is paying 5% annual effective. Which is paying more?
20. What is the annual effective rate on an investment that is paying 6% compounded quarterly?
21. What is the present value of a payment of \$12,000 to made at the end of 6 years if the interest rate is 7% effective?
22. What is the value in eight years (i.e. the future value) of a payment now of \$45,000 if the interest rate is 4.5% effective?
23. George is borrowing \$20,000 and will pay 8.5% interest . He will pay off the loan in three annual installments, \$6,000 at the end of the first year, \$7,000 at the end of the second year, and a final payment at the end of the third year. What should the amount of the final payment be?
24. Alice is saving money for a new car she hopes to buy in four years. She is putting the money in an account that earns 5.5% effective. At the beginning of this year, she deposited \$4,000, and the beginning of the second year, she expects to deposit \$5,000, and the beginning of the third year, she expects to deposit \$5,000 again. She anticipates the price of the car she wants to buy will be \$20,000. How much more will she need in four years to make the purchase?
25. Value of \$12500 invested for 3 years at 6% compounded quarterly.
26. Amount you need to invest now to have \$25000 in 4 years if your account pays 6% compounded monthly.
27. Amount you need to invest now to have \$100000 in 15 years if your account pays 5% compounded monthly.
28. You can buy a \$25 Series EE savings bond for \$18.75; that is, you invest \$18.75 today and in 6 years, you get back \$25. What is the annual effective interest rate that you are getting.

29. Assuming that house prices have inflated at an average rate of 8% per year for each of the last 20 years, how much would a house that is currently worth $100,000 have cost 20 years ago?
30. On Jan. 1, 1998, I opened an account in a bank yielding 3.4% annual (compound) interest. On each subsequent Jan. 1, I made either a deposit or a withdraw according to the following chart. What is my balance on Dec. 31, 2002?

Year	1998	1999	2000	2001	2002
De./With.	2000	−700	600	−200	1500

31. I borrow $5,000 at 7.1% compound interest per year for 5 years with yearly payments starting at the end of the first year . My first 4 payments were: $1,000, $700, $2,000, and $1,000. What is my last payment?
32. I want to be able to buy a $25,000 car in ten years. If I can invest money at 8% compound interest, how much do I need to invest now?
33. What is the present value of $25,000 ten years from now at 8% interest?
34. If I invest $25,000 now at 8% compound interest, how much will I have in ten years?
35. What is the future value of $25,000 ten years from now at 8% interest?
36. I buy a piano from Cheapside Music company on March 1, 1998. I pay $1,000 immediately, then 3 more payments of $1,000 on March 1 for each of the next 3 years. Finally, on March 1, 2002, I pay $10,000. What was this deal worth to Cheapside on March 1, 1998, assuming that they can invest funds at 4% interest–i.e. what was the present value of all of my payments on March 1, 1998?
37. In Exercise 36, assuming that Cheapside did invest all of my payments at 4% interest, how much did they have in their account on March 1, 2002?
38. You just won the Publisher Clearing House grand prize which is $1,000,000 paid in 10 annual installments of $100,000 each. Assuming that you can invest money at 3.9% compound interest, what is the present value of this prize?
39. An insurance company earns 7% on their investments. How much must they have on reserve on January 1, 2002 to cover the claims for the next 3 years, if they expect claims of $500,000 for 2002, $300,000 for 2003 and $250,000 for 2004. For sake of simplicity, assume that the claims are all paid on Dec. 31 of the stated year.
40. On January 1, 1995, Susan put $5,000 into a bank account at Stingy's Bank which pays 3.5% compounded twice a year. On July1, 1996, she withdrew $500. On January 1, 1997 she deposited an additional $700. How much did she have on account on January 1, 2000?
41. What is the effective annual rate of return on an investment that grows at a discount rate of 7%, compounded monthly for the fist two years and at a force of interest of 5% for the next 3 years?
42. An interest rate of 6% compounded three times a year is equivalent to what rate of interest compounded twice a year.
43. I have a bank account that initially has $6,000. After 2 years I withdraw $4,000. After 4 years, I empty the account by withdrawing $3,000. What is the annual effective interest rate i?

3. Rate of Return

In finance, one of the most fundamental problems is determining the rate of return on a given investment over a given year, the *annual effective rate of return.* In the simplest case, we put money in at the beginning of the year and don't touch it until the end of the year. In this case, the annual effective rate of return i is defined by

$$1 + i = \frac{B_1}{B_0} \tag{10}$$

where B_1 is the end balance and B_0 is the initial balance. Thus, the annual effective rate of return, as a percentage, is the percentage of growth in the account over the year. (Of course, it can also be negative, in which case it represents the percent of decrease in the account.)

The problem of determining the rate of return becomes more complicated when money is being added to or subtracted from the account throughout the year. This situation might arise, for example, in a large mutual fund in which individual investors periodically buy and sell shares. In this case, there are several different ways of defining the rate of return which do not necessarily yield the same answer. One of the most common is the following.

DEFINITION 4. *An* effective rate of return (ROR) *(or* dollar weighted rate of return *or* internal rate of return *or* yield rate*) on an investment over a given period of time is an interest rate i which would yield the same final balance for the same activity in the account. It is found by solving equation (5) for i.*

According to formula (9) the ROR can also be described as *that interest rate i for which the total present value at time 0 of all of the activity in the in the investment is 0, where the initial balance is considered as income and the final balance as outgo.*

Typically, finding an exact solution for i in equation (5) is impossible. However, for short periods of time (less than a year) the observation that there is little difference between simple and compound interest allows us to approximate the solution, as in the next example.

EXAMPLE 13. The following chart is a record of the activity in an investment. The initial balance was \$10,000 and the final balance was \$10,176.22. Approximate the annual effective rate of return.

Date	Deposit(+) or Withdraw(-)
Jan. 1	0
Apr. 1	1500
Sept. 1	-1000
Jan. 1	0

SOLUTION. In this case, equation (5) yields

$$10176.22 = (1 + i)10000 + (1 + i)^{9/12}1500 - (1 + i)^{4/12}1000$$

Replacing compound interest with simple interest yields

(11)

$$\begin{aligned}10176.22 &\approx (1+i)10000 + (1+\frac{3}{4}i)1500 - (1+\frac{i}{3})1000\\ 10176.22 - 10000 - 1500 + 1000 &\approx (10000 + \frac{3}{4}1500 - \frac{1}{3}1000)i\\ \frac{-323.78}{47166.67} &= -.030 \approx i\end{aligned}$$

Hence, the annual effective rate of return was approximately -3%.

The technique demonstrated in Example 13 is the standard method of computing the annual rate of return. In general, applying this technique to equation (5) with $t = 1$ yields the following formula, whose derivation is left as an exercise for the reader:

(12)
$$i \approx \frac{B(1) - (B(0) + C_1 + C_2 + \cdots + C_n)}{B(0) + (1-t_1)C_1 + (1-t_2)C_2 + \cdots + (1-t_n)C_n}$$

Remembering equation (12) is easy: the numerator is the final balance minus all of the year's activity in the account. Hence the numerator represents the growth of the account due to interest since, witout interest, this number would be 0. The denominator is the sum of the deposits weighted according to how long they were on deposit. *Equation (12) is very useful for actuarial exams and should be memorized.*

EXAMPLE 14. Use equation (12) to solve Example 13.

SOLUTION. Equation (12) yields

$$i \approx \frac{10176.22 - 10000 - 1500 + 1000}{10000 + \frac{3}{4}1500 - \frac{1}{3}1000} = -.030$$

as before.

It should be remarked that the BA II Plus can find a more accurate approximation to the dollar weighted ROR, which it calls the *internal rate of return (IRR).* See pp. 67-69 of the BA II manual for an example of such a calculation. This feature, however, is not as useful as one might expect; we find it easier to apply formula (12). If one does choose to use the BA II, some important points to remember are:

1. The example in the manual suggests resetting the calculator before beginning the computation. DON'T! Instead, after entering [CF], enter [2nd][CLR][WORK] which clears the current work sheet without resetting the defaults for the calculator.
2. The BA II expects equal periods between the cash flows. Hence, if the deposits are monthly, as in Example 13, *a value must be entered for each month of the year.* Sequences of months with 0 cash flow can be entered as one cash flow with an appropriate frequency, as in the example in the manual.
3. The computed ROR is *per period*, which in the case of Example 13, is per month. Hence, one would need to convert to a yearly rate by returning to calculator mode ([CE/C]) and computing $(1 + [2nd][Ans])^{12}$.

Occasionally, an exact answer can be found for the ROR can be found as in the next example.

EXAMPLE 15. Find the rate of return for an investment that yields the income stream indicated in the time line below.

SOLUTION. We need to solve the following equation for i:

$$50(1+i)^2 - 115(1+i) + 66 = 0$$

From the quadratic formula

$$\begin{aligned} 1+i &= \frac{115 \pm \sqrt{(115)^2 - 4 \cdot 66 \cdot 50}}{100} \\ &= \frac{115 \pm \sqrt{25}}{100} \\ &= \frac{115 \pm 5}{100} \end{aligned}$$

Hence $1+i$ is either 1.1 or 1.2, implying that the ROR is either 10% or 20%. There is no way of choosing between these answers; both are correct.

Example 15 makes an important point: the ROR might not be unique. This situation is rare, but does occur. There is also no guarantee that the ROR exists at all–i.e. for certain payment valued C_i, equation (5) might not have any solutions. For example, if the first payment in example 15been 51 instead of 50, then there are no real solutions to equation (5) since

$$(115)^2 - 4 \cdot 66 \cdot 51 = -239$$

The reason non-uniqueness is rare is that it is a theorem, which we shall not prove, which states that *if the balance in an account is never negative, then the ROR i, if it exists, is unique as long as it satisfies $i > -1$.*

Another situation under which the ROR is unique is *where the all of the payments to the account occur prior to any payments out of the account.*

Computing the dollar weighted rate of return requires knowing the amounts of the deposits/withdrawals and the times at which they occurred. There is another way of defining a rate of return that requires knowing instead the balances before and after each deposits/withdrawal. The idea is most easily demonstrated with an example. It is convenient to consider the initial balance as an initial deposit which opens the account and the final balance as a final withdrawal which closes the account.

EXAMPLE 16. The following is a record of the balances in an account which earned compound interest at rate i over a single year. Find i.

Activity	Balance before Activity	Balance after Activity
Deposit	0	50,000
Withdrawal	51,000	46,000
Deposit	46,500	47,500
Withdrawal	50,000	0

SOLUTION. Assume that the first activity occurred at time t_1 and the second at time t_2. Over time t_1, the account grew from 50,000 to 51,000. Hence

$$\begin{aligned} (1+i)^{t_1} 50000 &= 51000 \\ (1+i)^{t_1} &= \frac{51000}{50000} \end{aligned}$$

Over the time interval from t_1 to t_2, the account grew from 46,000 to 46,500. Hence

$$(1+i)^{t_2-t_1}46000 = 46500$$
$$(1+i)^{t_2-t_1} = \frac{46500}{46000}$$

Finally, over the time interval from t_2 to 1, the account grew from 47,500 to 50,000. Hence

$$(1+i)^{1-t_2}47500 = 50000$$
$$(1+i)^{1-t_2} = \frac{50000}{47500}$$

It follows that

$$\begin{aligned} 1+i &= (1+i)^{t_1}(1+i)^{t_2-t_1}(1+i)^{1-t_2} \\ &= \frac{51000}{50000}\frac{46500}{46000}\frac{50000}{47500} = 1.085354691 \end{aligned}$$

Hence, $i = 8.54\%$.

In general, suppose that over the space of a year, an account earning compound interest at rate i had n deposits/withdrawals occurring at times $t_0, t_1, \ldots t_n$. We consider the initial balance as an initial deposit and the final balance as a final withdrawal so that $t_0 = 0$ and $t_n = 1$. Let B'_k be the balance just before the kth transaction let B_k be the balance just after the kth transaction. Over the period from t_k to t_{k+1}, the account grew from B_k to B'_{k+1} Hence,

$$(1+i)^{t_{k+1}-t_k} = \frac{B'_{k+1}}{B_k}$$

Hence, as in Example 16,

$$1+i = \frac{B'_1}{B_0}\frac{B'_2}{B_1}\frac{B'_3}{B_2}\cdots\frac{B'_n}{B_{n-1}} \tag{13}$$

In words, formula (13) says that $1+i$ is product of the end balances for each time period divided by the product of the beginning balances for each time period.

If the account was an investment, such as a mutual fund, where the interest rate was not constant over the year, we can still use formula (13) to define an average rate of return for the account:

DEFINITION 5. *The time weighted rate of return on an account is the number i defined by formula (13) where B_0 and B'_n are, respectively, the initial and final balances in the account and for $0 < i < n$, B'_i and B_i are, respectively, the balances immediately before and after the ith transaction.*

In analyzing an investment, we can use either the time weighted or the dollar weighted methods to compute the rate of return. If the rate of return was relatively constant over the year, then both will yield similar answers. In fact, if the yield rate i is constant, then both methods yield exactly the same answer–i. In a very volatile investment, the two methods will vary as they emphasize different aspects of the investment. These issues are explored further in the exercises.

EXAMPLE 17. The following is a record of the balances in a fund over 2001. (a) Compute the time weighted rate of return and (b) approximate the dollar weighted rate of return.

Date	Activity	Amount	Balance before
Jan. 1	Deposit	50000	0
Mar. 1	Withdrawal	5000	46,000
July. 1	Deposit	1000	49,000
Jan. 1	Withdrawal	51000	51,000

Solution. From formula (12), the dollar weighted ROR is (approximately)

$$i = \frac{51000 - 1000 + 5000 - 50000}{50000 - 5000(10/12) + 1000(6/12)} = .1079136691$$

for a 10.8% return.

To facilitate computing the time weighted ROR we compute the beginning balances by adding the deposits to (subtracting the withdrawals from) the end balances:

Date	Activity	Amount	Balance before	Balance after
Jan. 1	Deposit	50000	0	50,000
Mar. 1	Withdrawal	5000	46,000	41,000
July. 1	Deposit	1000	49,000	50,000
Jan. 1	Withdrawal	51000	51,000	0

Hence, from formula (13)

$$1 + i = \frac{46000 \cdot 49000 \cdot 51000}{50000 \cdot 41000 \cdot 50000} = 1.121502439$$

for a 12.15% ROR.

A mutual fund is a organization which typically invests contributions from many different sources into many different investment vehicles. When an individual investor chooses to sell his/her share, the payment depends on the growth (or decline) of the fund a whole over the period of investment. Under the *portfolio method*, a yield rate for the fund as a whole is computed over the period of the investor's contributions and is used to determine the value of his/her account.

The portfolio method, however, is not always an accurate indication of the performance individual contributions. Suppose, for example, we can invest money at a higher rate now than we could have at the same date last year. Since the term on most investments is longer than a year, funds invested this year ("new money") make a greater contribution to the growth of the fund than funds invested last year. To award these funds only the average ROR of the mutual fund is over the past few years does not seem reasonable.

To account for such factors, many investment firms use the *investment year method*. We will describe it on an annual basis, although typically smaller time intervals would be used. When incremented annually, at the end of each year, a separate rate of return is declared for each possible year of investment. Thus, for example, a portion of the ROR chart for a fund might look something like the following:

Purchase Year		1994	1995	1996	1997	1998	1999	2000	2001	2002
1994	*Yields*	6.4%	6.8%	7.1%	6.9%	7.3%	5.1%	4.9%	4.8%	4.7%
1995	*Yields*		6.9%	7.0%	7.0%	7.4%	5.0%	4.9%	4.8%	4.7%
1996	*Yields*			7.1%	7.3%	7.3%	5.5%	5.4%	4.8%	4.7%
1997	*Yields*				7.0%	7.4%	5.4%	5.2%	4.6%	4.7%
1998	*Yields*					7.2%	5.7%	5.5%	4.5%	4.4%
1999	*Yields*						5.8%	5.1%	4.3%	4.7%
2000	*Yields*							5.0%	4.1%	4.6%
2001	*Yields*								4.0%	4.5%
2002	*Yields*									4.1%

ROR Chart

Thus, for example, if we had invested $1,000 on Jan. 1, 1994 and $500 on Jan. 1, 1995, then our total accumulation on Jan. 1, 2003 would be

$$(1.064)(1.068)(1.071)(1.069)(1.073) \\ \cdot (1.051)(1.049)(1.048)(1.047)1000 = 1688.75$$

from our 1995 contribution, plus

$$(1.069)(1.070)(1.070)(1.074) \\ \cdot (1.050)(1.049)(1.048)(1.047)500 = 794.31$$

from our 1996 contribution, for a total of

$$1688.75 + 794.31 = 2483.06.$$

Notice that from the ROR Chart, investments made between 1994-1996 all earned the same rate of return both in 2001 (4.8%) and 2002 (4.7%). This illustrates the principle that under the investment year method, funds on investment longer than a certain fixed period (5 years in our example) are assumed to grow at the overall ROR for the fund (the *portfolio rate*), regardless of when the funds were invested.

To better indicate the switch from the investment year method to the portfolio method, the data from the ROR chart above would typically be displayed in the following abbreviated format where the first five year's yield rates for each investment year are listed beside the year and the portfolio rates are listed in a separate column to the right. Thus, the investment rates for a given investment year are the entries in the row to the right of the year, including the entry in the Port Folio Rate column, together with all of the entries in the Portfolio column below the given year.

.

Purchase Year (y)	y	$y+1$	$y+2$	$y+3$	$y+4$	Portfolio Rate	Portfolio Year
1994	6.4%	6.8%	7.1%	6.9%	7.3%	5.1%	1999
1995	6.9%	7.0%	7.0%	7.4%	5.0%	4.9%	2000
1996	7.1%	7.3%	7.3%	5.5%	5.4%	4.8%	2001
1997	7.0%	7.4%	5.4%	5.2%	4.6%	4.7%	2002
1998	7.2%	5.7%	5.5%	4.5%	4.4%		
1999	5.8%	5.1%	4.3%	4.7%			
2000	5.0%	4.1%	4.6%				
2001	4.0%	4.5%					
2002	4.1%						

Abbreviated ROR Chart

For example, reading all of the way to the right, and then down, from 1995 yields the sequence

6.9% 7.0% 7.0% 7.4% 5.0% 4.9% 4.8% 4.7%

which is the row corresponding to 1995 in the ROR Chart

It should be commented that determining the entries for the ROR Cart is somewhat complicated. Suppose, for example, a security, bought in 1994, was sold in 1996. The income might then be used to purchase a new security. Do the returns on this new investment count as 1994 returns since the original investment was made in 1994, or do the returns count as 1996 returns since they came from a 1996 purchase? Under the first described approach, money that came into the fund in 1994 remains "1994 money" for ever (or rather until the portfolio rate kicks in). For this reason, this approach is called the *fixed index system.* If reinvested funds are associated with the reinvestment year, then the amount of "1994 money" will gradually decline. This approach is called the *declining index system.*

Another application of rates of return is to what are called *short sales.* Imagine that I run an investment house that holds 10 shares of stock in your name which is currently worth $100 per share. I anticipate that the value of this stock will decline in the coming year. I sell your stock at the beginning of the year. Suppose that at the end of the year, this stock is selling at $90 per share. I buy back your 10 shares for $900, and walk away with a $100 profit. You still have your stock and I have $100 profit. Selling securities that you don't own with the intention of eventually repurchasing them is called *selling short.*

At first glance, it appears that the yield rate on a short sell is infinite. In the above example, I earned $100 with no investment. In actuality, however, my yield rate was not infinite. If the value of your stock had gone up, I still would have had to buy it back whenever you demand it. If I wish to avoid going to jail, I need to have enough money in reserve, over and above the $1000 earned by selling your stock, to cover any potential gain in value of your stock. These reserves, which are called the *margin*, are my investment; they are what I put at risk. Typically the margin is described as a percentage of the value of the security sold and is fixed by law. Thus, in my example I might be required to have a margin of 50% of the value of the security, in which case my margin is \$500. Thus, I invested \$500 and earned \$100, making my yield rate 20%. I am allowed to invest the margin in some (presumably secure) investment. Hence, I would also earn interest on the margin, which would increase my yield rate slightly. I am (by law) not allowed to invest the original \$1,000 as this, in principal, is not my money.

Computing the yield rate on a short sell is slightly more complicated if the security sold yields dividends or income since I must pay all dividends and/or income out of my own funds, as if the security had never been sold.

EXAMPLE 18. On Jan. 1, I sell a bond short for \$5,000. My margin requirement is 40% and I invest the margin at 3% annual effective. The bond pays \$30, quarterly. At the end the fourth quarter, I repurchase the bond for \$4,700. What was my dollar weighted ROR?

SOLUTION. We use the quarter as our basic time interval. The profit on the short sale was \$300. Our margin was $.4 \cdot 5000 = 2000$. Over the year, our margin earned $.03 \cdot 2000 = 60$ dollars interest. Thus our year's income/outgo was

0	1	2	3	4
-2000	-60	-60	-60	$-60 + 2060 + 300$

Thus, the dollar weighted ROR is (approximately)

$$i = \frac{2300 - 60 - 60 - 60 - 2000}{2000 + \frac{3}{4}60 + \frac{1}{2}60 + \frac{1}{4}60} = .0574$$

for a 5.74% return.

4. Discount and Force of Interest

According to formula 7, the value of money decreases as we look backward in time. For example, at 5% interest, \$1,000 today was worth only

$$(1.05)^{-1}1000 = 952.38$$

last year. In terms of percentages, last year's value was 4.762% lower than this year's value. The 4.762% is the called the "rate of discount."

In general, if an account earns interest at rate i, the *rate of discount* is the number d defined by

$$1 - d = (1 + i)^{-1} \tag{14}$$

The number $100d$ is the *discount percentage* for the account. Just as in the above example, d represents the rate of decrease in value of the account as we look backwards in time. Specifically, P today was worth

$$(1 + i)^{-1}P = (1 - d)P = P - dP \tag{15}$$

last year, showing that last year's value was a factor of d lower than this year's value.

EXAMPLE 19. What is the annual effective rate of interest for an account that earns interest at the discount rate of 3.7%.

SOLUTION. According to formula (15), the interest rate is

$$(1 - .037)^{-1} = 1.0384$$

making the interest rate 3.84%.

Just as with interest, there are also nominal discount rates. If we speak, say, of an account having a discount rate of 3% compounded monthly, we are saying that each month's discount is 3/12%.

DEFINITION 6. *An account grows at a discount rate of d compounded n times per year if the discount rate per compounding period is $\frac{d}{n}$. We use the symbol "$d^{(n)}$" to indicate that d is compounded n times per year.*

EXAMPLE 20. What is the annual effective rate of interest on an account with a discount rate of $.03^{(12)}$? What is the interest rate stated as a nominal rate, compounded monthly?

SOLUTION. Our monthly discount rate is .03/12. Thus, from formula (15), the monthly interest rate is i where

$$1 + i = (1 - .03/12)^{-1} = 1.002506.$$

The annual effective interest rate j satisfies

$$1 + j = (1.002506)^{12} = 1.0305$$

making the annual effective rate of interest rate 3.05%.

The nominal monthly rate is 12 times the monthly rate–i.e. $12 \cdot (.002506) = .03072$ which, as a percent, is 3.072%.

EXAMPLE 21. How much will we have after 10 years if we invest \$ 2000 at a discount rate of 4% per year, compounded monthly.

SOLUTION. From Formula 15 the monthly interest rate is i where

$$1 + i = (1 - \frac{.04}{12})^{-1} = 1.003344482$$

Hence $i = .003344482$. Thus, using Formula 6, our accumulation is

$$(1.003344482)^{120} 2000 = 2985.64$$

dollars.

Another common measure of interest is what is called the "force of interest." From Formula 6, the balance $B(t)$ in the account at time t satisfies

$$\begin{aligned} B(t) &= (1+i)^t P \\ &= e^{\ln((1+i)^t)} P \\ &= e^{t\delta} P \end{aligned} \tag{16}$$

where

$$\delta = \ln(1+i) \tag{17}$$

The number δ is the *force of interest.*

EXAMPLE 22. An account grows with a force of interest of .0334 per year. What is the interest rate?

SOLUTION. From Formula 17,

$$1 + i = e^{\delta} = e^{.0334} = 1.034998523$$

Thus, $i = .035$.

Note that differentiating Formula 16 produces

$$B'(t) = \delta B(t)$$

This tells us two important things:

1. The rate of growth of the amount function is proportional to the amount of money in the account.
2. The proportionality constant is the force of interest.

We can use the force of interest to describe circumstances in which the interest rate varies over time. Specifically, solving the preceding equation for δ produces

$$\delta = \frac{B'(t)}{B(t)} = \frac{d}{dt}(\ln B(t)) \tag{18}$$

DEFINITION 7. *If $B(t)$ represents the amount in an account at time t, then the force of interest for this account is the quantity $\delta(t)$ defined by Formula 18.*

Note that from Formula 18

$$\begin{aligned}\int_{t_0}^{t_1} \delta(t)\, dt &= \ln B(t_1) - \ln B(t_0) \\ &= \ln \frac{B(t_1)}{B(t_0)}\end{aligned}$$

This formula proves the following proposition.

PROPOSITION 1. *If an account grows with force of interest $\delta(t)$ over the interval $[t_0, t_1]$, then*

$$B(t_1) = e^{\int_{t_0}^{t_1} \delta(t)\, dt} B(t_0) \tag{19}$$

EXAMPLE 23. What is the annual effective rate of return on an account that grew at 4% interest per year for the first 2 years, a force of interest of $\delta_t = \frac{1}{2+t}$ for the next 3 years, and a discount rate of 4% for the last 2 years?

SOLUTION. From formulas (19) and (15) over the 7 year period P dollars will grow to

$$\begin{aligned}&(1 - .04)^{-2} e^{\int_2^5 \frac{1}{2+t}\, dt}(1.04)^2 P \\ &= (1.085)e^{(\ln 7 - \ln 4)}(1.0816)P \\ &= (1.085)(\frac{7}{4})(1.0816)P \\ &= (2.054)P\end{aligned}$$

The annual effective rate i is determined by solving the equation

$$(1+i)^7 = 2.054$$

which yields $i = 10.82\%$.

5. Annuities

The single most important theorem in interest theory is

$$x^n + x^{n-1} + \cdots + x + 1 = \frac{x^{n+1} - 1}{x - 1} \tag{20}$$

The proof is simple:

$$\begin{aligned}(x^n + x^{n-1} + \cdots + x + 1)(x-1) &= (x^n + x^{n-1} + \cdots + x + 1)x - (1 + x + x^2 + \cdots + x^n)1 \\ &= x^{n+1} + x^n + \cdots + x^2 + x \\ &\quad - x^n - x^{n-1} - \cdots - x - 1 = x^{n+1} - 1\end{aligned}$$

which is equivalent with formula (20).

The significance of formula (20) is that it is used to analyze annuities. An *annuity* is an account into which we make either periodic deposits or periodic withdrawals. If the transactions always occur at the end of the compounding period, the annuity is said to be an *annuity immediate* while if the transactions always occur at the beginning of the compounding period, the annuity is said to be an *annuity due*. If we deposit C dollars at the end of the year for each of n years into an account that earns compound interest at rate i, then, from formulas (5) and (20), our balance is

$$B(n) = (1+i)^{n-1}C + (1+i)^{n-2}C + \cdots + (1+i)C + C = \frac{(1+i)^n - 1}{i}C,$$

which we may express as

$$B(n) = s_{\overline{n}|i}C$$

where

(21) $$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

EXAMPLE 24. I deposited \$300 at the end of each year from 1981 to 2000 into an account that yields 3% interest per year. How much do I have at the end of 2000?

SOLUTION. From formula (21) I have

$$300 s_{\overline{20}|.03} = 300\frac{(1.03)^{20} - 1}{.03} = 8061.11$$

dollars.

On the BA II Plus we enter [2nd][P/Y]1,[ENTER], [CE/C], 3[I/Y], 0[PV],20[N], 300[PMT], [CPT][FV], [+/-]yielding 8061.11 . If there had been an initial balance in the account, we would have entered it (as a positive value) into the PV register.

Depositing C at the beginning of each year is equivalent with depositing $(1+i)C$ at the end of the year. Hence, n deposits of C at the beginning of each year accumulates to

$$B(n) = (1+i)s_{\overline{n}|i}C.$$

The corresponding accumulation function is

(22) $$\ddot{s}_{\overline{n}|i} = (1+i)s_{\overline{n}|i}$$

EXAMPLE 25. I deposited \$300 at the *beginning* of each year from 1981 to 2000 into an account that yields 3% interest per year. How much do I have at the end of 2000?

SOLUTION. From formula (22) we have

$$(1.03)s_{\overline{20}|.03}300 = (1.03)8061.11 = 8302.95$$

dollars.

On the BA II, we first set the calculator to beginning mode by entering [2nd][BEG][2nd][SET]. We then enter the data just as in Example 24. Don't forget to reset the calculator to END mode after the calculation is finished.

We may use formulas (21) and (22) for accounts where the periods between deposits differs from the compounding period. We only need to remember that i is the interest per deposit period and n is the number of deposit periods.

Example 26. At the beginning of 1992, I opened a bank account earning 4% interest, compounded quarterly, with a \$5,000 deposit. I deposited \$100 at the beginning of each month from 1992 to 2001. What was my balance on Dec. 31, 2001?

Solution. Over the 10 years from Jan. 1, 1992 to Dec. 31, 2001, the original \$5000 grew to

$$(1 + .04/4)^{40}5000 = 7444.32$$

The annual effective interest rate is $(1 + .04/4)^4$. Hence, each month, my account grows by

$$(1.04/4)^{4/12} = 1.003322284$$

making the monthly rate .003322284. Ten years is the same as 120 months. Hence, from formula (22), the monthly deposits accumulated to

$$1.003322284 \frac{(1.003322284)^{120} - 1}{.003322284} 100 = 14763.58$$

Hence, the total is

$$7401.22 + 14717.61 = 22118.83$$

dollars.

On the BA II Plus, after setting th calculator to beginning mode, we would enter [2nd][P/Y]. The display should read "$P/Y =$". Enter 12[ENTER] (12 payments/year), followed by the down (or up) arrow. The display now should read "$C/Y =$". Enter 4[ENTER] (quarterly compounding)[CE/C]. We would then enter -5000[PV], -100[PMT], 10[2nd][xP/Y][N], [CPT][FV].

Remark. Computations using short compounding periods over long periods of time are highly susceptible to round-off error. For example, had we rounded the monthly rate from Example (26) to .003, then we would compute the accumulation of the monthly deposits as

$$(1.003) \frac{(1.003)^{120} - 1}{.003} 100 = 14461.83$$

which is off by almost \$300! When doing compound interest problems without a financial calculator, you should make full use of the memory of your calculator, writing as little on paper as possible. Specifically, in Example (26), I would compute the interest rate as $(1.01)^{1/3} - 1$, never recording the answer. I would compute the final answer as

$$[2nd][Ans] * \frac{(1 + [2nd][Ans])^{120} - 1}{[2nd][Ans]} 100 = 14763.58$$

As mentioned previously, in actuarial work, the most typical example of an annuity is a retirement account where the individual accumulates a sum of money while employed, intending to make periodic withdrawals over a space of time to cover living expenses. It follows from the prospective method (formula (9)) that *the balance required at time 0 to make future payments of* C_1, $C_2, \ldots, C_n$ *at times* t_1, $t_2, \ldots, t_n$ *is the sum of the present values at time 0 of the payments.* In particular, to receive n payments of C at the end of each compounding period, i.e. for an annuity immediate, our initial balance must be

$$\begin{aligned} B &= (1+i)^{-n} s_{\overline{n}|i} C \\ &= a_{\overline{n}|i} C \end{aligned}$$

where

$$a_{\overline{n}|i} = (1+i)^{-n} \frac{(1+i)^n - 1}{i} = \frac{1-\nu^n}{i} \tag{23}$$

If we wish to receive the payments at the beginning of the compounding period, i.e. for an annuity due, we require an initial balance of

$$\begin{aligned} B &= (1+i) a_{\overline{n}|i} C \\ &= \ddot{a}_{\overline{n}|i} C \end{aligned}$$

where

$$\ddot{a}_{\overline{n}|i} = (1+i) a_{\overline{n}|i} \tag{24}$$

EXAMPLE 27. I Plan to retire at age 70, at which time I will withdraw \$5,000 per month for 20 years from my IRA. I also want to receive a final payment of \$8,000 for a 90th birthday trip to Hawaii. Assuming that my funds are invested at 4.7% interest, compounded monthly, how much must I have accumulated in my IRA?

SOLUTION. The interest rate is $i = .047/12$ and $\nu = (1+i)^{-1} = .9960986134$. The required balance is the present value of the future payments which, from the above discussion, is

$$\begin{aligned} &a_{\overline{20\cdot 12}|i} 5000 + \nu^{20\cdot 12} 8000 \\ &= \frac{1-\nu^{240}}{i} 5000 + \nu^{240} 8000 \\ &= 777005.53 + 3130.77 = 780136.30 \end{aligned}$$

dollars.

On the BA II, I would enter [2nd][P/Y]12,[CE/C], 20[2nd][xP/Y], [N], −8000[FV], −5000[PMT], [CPT][PV]. Note that the payments are negative since we are withdrawing funds from the account.

REMARK. At times, figuring out what to enter into the calculator can be confusing, Suppose, for example, we wish to find the present value of 75 payments of \$200 at 6% interest per payment. It is clear that we can set $P/Y = 1$ and then enter 75[N], 200[PMT], 6[I/Y]. What is, perhaps, less clear is that we should enter 0[FV]. This might seems wrong at first because certainly, the future value of the payments is not 0! Remember, however, that the present value is the amount we need to invest today to eventually grow to the same amount as the total accumulation of the payments. Thus, in addition to receiving the payments, we are also making an investment at time 0–i.e. receiving a negative payment. The total future value is zero when we include this negative value.

A *perpetuity* is an annuity that generates a stream of income that lasts "forever." Just as with annuities, a *perpetuity immediate* generates income at the end of each compounding period while a *perpetuity due* generates income at the beginning of each compounding period. For example, an account that funds a $5,000 annual prize to be awarded at the end of each year could be considered as a perpetuity immediate.

In the typical perpetuity immediate, a sum P is invested at rate i. At the end of each payment period, the perpetuity pays out the interest earned during that period so that the principal remains intact. Thus, the payment is

$$C = iP.$$

Conversely, to receive payments of C at the end of each compounding period in perpetuity, then, at interest rate i, the principal must be

$$P = \frac{C}{i}.$$

If we wish to pay C at the beginning of each compounding period, then, to have the principal intact at the end of the period, we require

$$P = (1+i)(P-C)$$

which is easily solved to yield

$$P = (1+i)\frac{C}{i}.$$

We have seen previously that the principal required to make a finite series of payments in the future is the total present value of all of the payments. This suggests the following definitions.

DEFINITION 8. *The present value at interest rate i of a perpetuity immediate that pays C at the end of each compounding period is*

$$PV = \frac{C}{i}. \tag{25}$$

For an annuity due that pays C at the beginning of each compounding period, the present value is

$$PV = (1+i)\frac{C}{i}. \tag{26}$$

Another way of motivating this definition is to let n tend to infinity in formula (23):

$$\lim_{n\to\infty} a_{\overline{n}|i}C = \lim_{n\to\infty} \frac{1-\nu^n}{i}C = \frac{C}{i}.$$

Similarly,

$$\lim_{n\to\infty} \ddot{a}_{\overline{n}|i}C = \lim_{n\to\infty}(1+i)a_{\overline{n}|i}C = (1+i)\frac{C}{i}.$$

EXAMPLE 28. Purdue Life is interested in acquiring a company (IU Guarantee) that is expected to yield a net income of $400,000 at the end of each year after the acquisition for the conceivable future. If Purdue Life can invest funds at 8% interest per year, what is the most they should pay for this company?

SOLUTION. From formula (25), the present value of the future payments at 8% interest is

$$400000/.08 = 5,000,000$$

dollars, which is the most they should pay.

Another way of reasoning would be to note that \$5,000,000, invested at 8% per year would yield the same income stream, so this is the most the company is worth.

If the income had begun immediately, instead of at the end of the year, then, from formula (25), the final answer would be multiplied by 1.03.

The annuities discussed so far have been *constant* annuities in that the payments are the same each period. In a *decreasing* annuity immediate, at the end of each period, each payment is P less than the previous payment, where P is constant. For example, if $P = 10$, our payments might be 200, 190, 180, 170, ... , 110. As the reader will see shortly, such annuities may all be analyzed in terms of the special case where the last payment is P, in which case the first payment would be nP where n is the number of payments. In this case, if payments P are made at the end of each period into an account at interest rate i per period, the amount is $PDs_{\overline{n}|i}$ where

$$Ds_{\overline{n}|i} = n(1+i)^{n-1} + (n-1)(1+i)^{n-2} + \cdots + 2(1+i) + 1$$

To obtain a formula for $Ds_{\overline{n}|i}$, we differentiate both sides of formula (20) with respect to x, obtaining:

$$\begin{aligned} nx^{n-1} + \cdots + 3x^2 + 2x + 1 &= \frac{(n+1)x^n(x-1) - (x^{n+1}-1)}{(x-1)^2} \\ &= \frac{nx^n(x-1)}{(x-1)^2} + \frac{x^n(x-1) - (x^{n+1}-1)}{(x-1)^2} \\ &= \frac{1}{x-1}(nx^n) - \frac{1}{x-1}\frac{x^n-1}{x-1} \end{aligned}$$

Setting $x = 1 + i$ yields

(27) $$Ds_{\overline{n}|i} = \frac{n}{i}(1+i)^n - \frac{1}{i}s_{\overline{n}|i}$$

Note that

$$PDs_{\overline{n}|i} = \frac{Pn}{i}(1+i)^n - \frac{P}{i}s_{\overline{n}|i}$$

This formula says that *an n payment decreasing annuity with payments $nP, (n-1)P, \ldots, P$ has the same value as an n term annuity with initial balance nP/i and constant payment $-P/i$.* In using this formula, the reader must be careful to note that *i is the interest per compounding period; not the nominal interest rate.*

Example 29. We pay \$7,000 at the end of the first quarter into an account that earns interest at 3.3% compounded quarterly. Each quarter thereafter we deposit \$1000 less. What is the accumulation at the end of the year?

Solution. Or deposits were

7000 6000 5000 4000

which cannot be analyzed using formula (27) since the last payment was not 1000. However, the above payment stream is equivalent with paying

$$3000 + 4000 \quad 3000 + 3000 \quad 3000 + 2000 \quad 3000 + 1000$$

From the comments following formula (27) our annuity is equivalent with a constant annuity with initial balance

$$4 \cdot 1000/(.033/4) = 484848.4848$$

and payment

$$-1000/(.033/4) + 3000 = -118212.1212$$

Hence, the answer is

$$484848.4848(1+i)^4 - 118212.1212 s_{\overline{4}|i}$$

where $i = .033/4$. To compute this on the BA II, we enter

[2nd][P/Y]4[ENTER][CE/C]

4[N]

3.3[I/Y]

$4 * 1000/(.033/4) =$ [PV]

$-1000/(.033/4) + 3000 =$[PMT]

[CPT][FV]

[+/-]

getting 22315.34162.

An *increasing annuity immediate* is an annuity immediate with the property that each payment is P more than the previous payment, where P is constant. We don't need any new formulas to analyze increasing annuities, as the next example shows.

EXAMPLE 30. We pay \$4,000 at the end of the first quarter into an account that earns interest at 3.3% compounded quarterly. Each quarter thereafter we deposit \$1000 more. What is the accumulation at the end of the year?

SOLUTION. Or deposits were

$$4000 \quad 5000 \quad 6000 \quad 7000$$

which we interpret as

$$8000 - 4000 \quad 8000 - 3000 \quad 8000 - 2000 \quad 8000 - 1000$$

Hence, letting $i = .033/4$, we see that the accumulation is

$$8000 s_{\overline{4}|i} - 1000 D s_{\overline{4}|i}$$

which we compute by setting $PMT = 8000{+}1000/(.033/4)$, $PV = -4{\cdot}1000/(.033/4)$, and proceeding otherwise as in the preceding example, getting $22,232.159$.

The above example makes it clear that we do not need a separate formula for increasing annuities. It is, nevertheless, useful to have one, especially if one is, say, taking an actuarial exam. We leave it as an exercise to show that accumulation of an annunity immediate whose payments at the end of each payment period of $1, 2, \ldots, n$ is

$$Is_{\overline{n}|i} = \frac{1+i}{i}s_{\overline{n}|i} - \frac{n}{i} \tag{28}$$

This formula says that *the future value of a n payment increasing annuity with first payment P equals $-Pn/i$ plus the future value of an n term annuity with present value* 0 *and constant payment* $(1+i)P/i$ where, again, *i is the interest per compounding period; not the nominal interest rate.*

Example 31. Solve Example 30 using formula (28).

Solution. We interpret our deposits as

$$3000 + 1000 \quad 3000 + 2000 \quad 3000 + 3000 \quad 3000 + 4000$$

The accumulation is

$$3000s_{\overline{i}|4} + 1000Is_{\overline{i}|4}$$

where $i = .033/4$. To compute this, after having set i and P/Y as before, we enter

0[PV]

$3000 + (1 + .033/4)1000/(.033/4) =$[PMT]

[CPT][FV]

[+/-]

$-1000 * 4/(.033/4) =$

getting $22,232.159$ as before.

The present values of increasing and decreasing annuities are, of course, also important. These are denoted, respectively, by

$$Ia_{\overline{s}|i} = \nu^{-s}Is_{\overline{i}|a}$$
$$Da_{\overline{s}|i} = \nu^{-s}Ds_{\overline{i}|a}$$

Example 32. Compute the present value of the annuity in Example 29.

Solution. As in Example 29, our annuity is equivalent with a constant annuity with initial balance

$$4 \cdot 1000/(.033/4) = 484848.4848$$

and payment

$$-1000/(.033/4) + 3000 = -118212.1212$$

Our strategy is to first compute the present value of the payments and then add the initial balance. For this, if the calculator is still set as in Example 29, we can enter 0[FV], [CPT][PV],[+/-],[2nd][Ans]+484848.4848 =. Note that we needed to use the [+/-] key since, as we have noted previously, the computed present value is really the negative of the true present value. We get 21,593.9388.

Exercises

1. What is the amount you will have if you invest \$75 at the end of each month for 10 years if the account pays 7.5% compounded monthly?
2. How much will you will have on December 31, 2000 if you invested \$150 at the end of each month starting in January 1996 in an account that pays 4.8% compounded monthly?
3. What is the amount you can borrow today if you are willing to pay \$300 at the end of each month for 5 years for a loan that charges 9% (compounded monthly).
4. What is the amount you need to invest at the end of each month to have saved \$8000 at the end of 4 years if the account pays 5% compounded monthly?
5. You intend to start depositing \$100 into an account at the end of each month starting December 31, 2000. How long will it take you to save \$38,000 for your dream car if the account pays 6% compounded monthly?
6. You intend to start depositing \$200 into an account at the end of each month starting December 31, 2000. How long will it take you to save \$38,000 for your dream car if the account pays 6% compounded monthly? (Compare with previous problem.)
7. Starting January 31, 1990, you saved \$100 at the end of each month in an account that paid 6% compounded monthly. Starting August 31, 1999, you increased the monthly contribution to \$150. How much will you have accumulated by December 31, 2000?

6. Loans

In interest theory, the difference between borrowing money and saving money is only in the point of view. In a bank account, each deposit is in essence a loan to the bank and each withdrawal a partial repayment of the loan. The interest I earn on the account is the interest the bank pays me on this loan. Thus, the only difference between a bank loan and a bank account is in who is doing the lending and who is doing the borrowing. It follows that formula (5) also describes the balance on a loan after n payments where B_0 is the amount borrowed (the principal), C_i is the amount of the ith payment which is made at time t_i (C_i will be negative in this case), and i is the interest rate.

The most common method of repaying a loan is by equal, periodic payments of C, made at the end of each compounding period together, if necessary, with a final, additional, "balloon payment" made at the end of the term. This method of repayment is called *amortization.* In this case, from formulas (5) and (20), the balance after the nth payment is

$$B(n) = P(1+i)^n - Cs_{\overline{n}|i}. \tag{29}$$

EXAMPLE 33. I borrow \$25,000 to buy a car on which I pay \$1000 down and make monthly payments at the end of the month over the next 5 years. If I pay 7% interest, compounded monthly, what are my monthly payments?

SOLUTION. After my down payment, I owe \$24,000. Saying that the interest is 7% compounded monthly, means that the monthly interest rate is .07/12. From formula (29)

$$
\begin{aligned}
0 &= (1+\frac{.07}{12})^{5\cdot 12}24000 - \frac{(1+\frac{.07}{12})^{12\cdot 5}-1}{.07/12}D \\
0 &= 34023.01 - 71.59D \\
D &= 475.23
\end{aligned}
$$

which is our monthly payment.

On the BAII, we would simply enter 24000[PV], -200[PMT], 0[FV], [2nd][P/Y]12, [CE/C], 5[2nd][xP/Y][N], [CPT][PMT]].

Example 34. I make the following deal with a piano rental company. For $200 a month, I can rent a piano which is worth $15,000. After 10 years, I own the piano. In essence, they are loaning me $15,000 which I repay in installments of $200/month. What annual interest rate, compounded monthly, are they charging me for this loan?

Solution. Let i be the monthly interest rate. From formula (29), for my loan to be paid off in 10 years

$$0 = (1+i)^{120}15000 - (((1+i)^{120}-1)/i)200 \tag{30}$$

There is no direct way to solve this equation for i. On the BAII, we would simply enter 15000[PV], -200[PMT], 0[FV], [2nd][P/Y]12, [CE/C],10[2nd][xP/Y][N] [CPT][I/Y].

If we borrow P dollars at rate i per period, then at the end of one period we owe $P+iP$. If we pay only the interest iP, then we still owe P at the end of the period. Hence, *to pay off the loan with a finite number of equal payments, each payment must be greater than* iP. It is possible to have a loan where the payments are less than iP. In this case the lender will require that the borrower will pay off the remaining balance after a fixed period of time.

Definition 9. *An amount P lent at rate i per period is said to be lent at a premium if the periodic payment C satisfies $C > iP$. If $C \leq iP$ the loan is said to made at a discount.*

Example 35. I borrow $8,000 at 5% interest, compounded quarterly, on which I pay $75 per quarter for 10 years. What is my final, additional, payment?

Solution. Notice that $(.05/4)8000 = 100 > 75$. Hence our loan is at a discount, so our final payment will be larger than $8,000. Specifically, the final payment will be

$$8000(1+.05/4)^{40} - 75s_{\overline{40}|.05/4} = 9287.24$$

On the BAII, we would enter 8000[PV], -75[PMT], [2nd][P/Y]4, [CE/C], 5[I/Y], [CPT][FV].

For a borrower, an important question is "How much do I owe after the nth payment?" The prospective method tells us that *the outstanding balance is the present value of all of the remaining payments*. This method is useful when we are not given the original amount borrowed.

Example 36. I have a 30 year loan at 7.3% compounded quarterly on which I pay $500 each quarter together with an additional final payment of $3000. How much do I still owe at the end of 5 years? How much did I originally borrow?

SOLUTION. The present value of the $25 \cdot 4 = 100$ remaining payments is

$$3000(1+.073/4)^{-100} + a_{\overline{100}|.073/4}500$$

which we compute on the BAII: -3000[FV], -500[PMT], 7.3[I/Y], [2nd][P/Y]4[ENTER], [CE/C], 25[2nd][xP/Y][N], [CPT] [PV] finding that we still owe $23,398.74. The original amount borrowed is found by entering 30[2nd][xP/Y][N], [CPT][PV], getting $24,612.35.

Let $B(n)$ denotes the balance on a loan at interest rate i after the nth payment. The absolute value of the change in the balance

$$P(n) = |B(n) - B(n-1)|$$

is the *principal adjustment* in the nth period. Then

$$B(n) = B(n-1) \pm P(n)$$

with "$\pm$" depending on the sign of $B(n) - B(n-1)$.

Specifically, for a loan at a premium, $B(n) < B(n-1)$ and the minus sign holds. The principal adjustment is then less than the payment. For a loan at a discount $B(n) \geq B(n-1)$ and the plus sign holds.

To find a formula for $P(n)$, note that just before the nth payment, we owe $(1+i)B(n-1)$. Hence, immediately after the nth payment, we owe

$$B(n) = (1+i)B(n-1) - C \tag{31}$$

where C is our payment. Thus

$$P(n) = |(1+i)B(n-1) - C - B(n-1)| = |iB(n-1) - C| \tag{32}$$

Note that

For a loan at a premium, $iB(n-1) < C$ so $P(n) = C - iB(n-1)$ so

$$C = iB(n-1) + P(n)$$

We think of $iB(n-1)$ as that part of the payment C going into paying the interest on the loan and $P(n)$ as that part of the payment going into paying the principal.

EXAMPLE 37. In Example 36, divide the 21st payment into the portion going into paying the interest and that going into paying the principal.

SOLUTION. From the solution to Example 36, the balance after the 20th payment is $23,398.74. One quarter's interest on this balance is

$$(.073/4)23398.74 = 427.03$$

which is the portion of the payment going into interest. The rest of the payment, $500 - 427.03 = 72.97$, goes into paying the principal.

On the BAII, enter all of the data just as in Example 36. Then enter [2nd][AMORT] to put the calculator into the amortization work sheet. At the "$P1 =$" prompt press 21[ENTER]. Then press the down arrow to get the "$P2 =$" prompt. Press 21[ENTER]. Then press the down arrow; you see the balance after the 21st payment (32,325.77). Press the down arrow again; you see -72.293, the total principal paid between payments P1 (21 in our case) and P2 (again, 21). Press the down arrow again; you see -427.03, the total interest paid between payments P1 and P2.

If you would like to see what portion of the 22nd payment is interest and what is principal, press the down arrow again to return to the "$P1 =$" prompt and press [CPT] which will increase both P1 and P2 by one unit.

We can also solve equation(31) for $B(n-1)$:

$$B(n-1) = (1+i)^{-1}(B(n)+C).$$

Hence

$$P(n) = |B(n) - (1+i)^{-1}(B(n)+C)|$$

which simplifies to

$$P(n) = (1+i)^{-1}|iB(n)+C| = \nu|iB(n)+C| \tag{33}$$

An *amortization schedule* is a table which lists, payment-by-payment, the amount of the payment, the portion of the payment going into interest, the portion going into principal, and the balance. It is clear from the discussion at the end of Example 37 that we can generate an amortization schedule using the BAII. However, it is easier to use Excel. Below is the first 5 lines of an amortization schedule for the loan described in Example 36 through the 21st payment.

Loan Amt.	$24,612.35	%i	0.073	quarterly
Pmt. #	Value	Prin.	Int.	Balance
1	500	50.82	449.18	24,561.53
2	500	51.75	448.25	24,509.77
3	500	52.70	447.30	24,457.08
4	500	53.66	446.34	24,403.42
5	500	54.64	445.36	24,348.78

As the loan is paid off, the principal adjustment steadily increases. In fact, from formula (31),

$$\begin{aligned} B(n) - B(n-1) &= (1+i)B(n-1) - C - [(1+i)B(n-2) - C] \\ &= (1+i)(B(n-1) - B(n-2)) \end{aligned}$$

Hence

$$P(n) = (1+i)P(n-1)$$

Thus, $P(n)$ grows according to the compound interest formula. In particular, for all m and n

$$P(n) = (1+i)^{n-m}P(m) \tag{34}$$

REMARK. *Equations 34 and 32 are useful on actuarial exams and should be memorized.*

EXAMPLE 38. In the 17th payment on a loan, $125 goes into the principal, while in the 20th payment, $150 goes into principal. What is the interest rate on the loan? If each payment is $200, and the loan is at a premium, what was the original amount of the loan? Answer the same question under the assumption that the loan is at a discount.

SOLUTION. From equation (34), with $n = 20$ and $m = 17$,

$$\begin{aligned} 150 &= (1+i)^{20-17}125 \\ \frac{150}{125} &= (1+i)^3 \\ 1+i &= 1.062658569 \end{aligned}$$

for a 6.3% interest rate. It also follows from equation (34), with $n = 17$ and $m = 1$, and from equation (32) that

$$\begin{aligned}(1.062658569)^{16}|iB(0) - C| &= 125\\ |iB(0) - C| &= 47.27265944\end{aligned}$$

For a loan at a premium, $C > iB(0)$ so

$$47.27265944 = 200 - (.062658569)B(0)$$

which may be solved for $B(0)$ yielding $B(0) = 2,437.45$ which is the original amount of the loan.

For a loan at a discount, the interest is greater than or equal to the payment so

$$47.27265944 = (.062658569)B(0) - 200$$

which yields $B(0) = 3,946.35$.

To find the balance after the 20th payment, we apply payment,

If each payment on a loan at interest rate i is $C = iP$ where P is the principal, then the balance after each payment is P. Thus, we will need to pay P at the term of the loan. To get the money to pay this final payment, we could make regular deposits into an account at, perhaps, a different rate of interest, say j. Such an account is called a *sinking fund* since as its value grows, the amount we need pay out of our pocket to make the final payment on our loan sinks. Typically the payments to the sinking fund will be made simultaneously with the payments on the original loan.

EXAMPLE 39. I borrow $350,000 from a bank at 5.5% compounded monthly which I will repay in 13 years by making monthly payments into a sinking fund owned by the bank that earns interest at 4.4%, compounded monthly. What are my total monthly payments? What is the bank's annual effective rate of return on its investment?

SOLUTION. Our payments C into the sinking fund must total to $350,000. Hence, for $i = .044/12$,

$$Cs_{\overline{13\cdot 12}|i} = 350000$$

which can be solved for C. In reality, however, we use the BAII setting [2nd][P/Y]12[ENTER], [CE/C], 5.5[I/Y],13[2nd][xP/Y][N], 0[PV],-350000[FV],[CPT][PMT]. Our monthly payment into the sinking fund is $1,666,76.

We also must pay the monthly interest on the $350,000 which is

$$(.055/12)350000 = 1604.17$$

making our total payment

$$1666,76 + 1604.17 = 3270.93$$

To find the bank's rate of return, we note that the bank pays out $350,000 and receives $13 \cdot 12 = 156$ payments of $3,270.93. The future value of he income must equal the future value of the outgo. Hence, the monthly ROR is found by solving

$$s_{\overline{156}|j}3270.93 - 350000(1 + j)^{156} = 0$$

for j. We would still, however, need to convert to annual compounding by setting

$$1 + k = (1 + j)^{12}$$

since an annual rate of return was requested.

On the BAII, we set [2nd][P/Y]12[ENTER] , [CE/C] , 156[N], -350000[PV], 3270.93 [PMT], 0[FV], [CPT][I/Y]. We find $i = 5.84\%$ annual ROR. This, however, is a nominal rate, compounded monthly. To convert to a yearly rate, we compute

$$(1+ \text{[RCL][I/Y]}/(100*12))[y^x]12 = 1.006090$$

for a 6.1% annual ROR. This is also our annual effective interest rate on the loan.

In general, if you borrow P at rate i per repayment period and pay into a sinking fund which earns rate j, then the payment into the sinking fund is

$$C = P/s_{\overline{N}|j}$$

where N is the total number of payments. Thus, the total payment is

$$\begin{aligned} C_{tot} &= iP + P/s_{\overline{N}|j} \\ &= \frac{is_{\overline{N}|j} + 1}{s_{\overline{N}|j}} P \end{aligned} \tag{35}$$

Suppose $i = j$. Then the numerator becomes

$$js_{\overline{N}|j} + 1 = (1+j)^N - 1 + 1 = (1+j)^N$$

so equation (35) is equivalent with

$$Cs_{\overline{N}|j} = (1+j)^N P.$$

From equation (29), it follows that C is also the payment required to pay off P in N years using the amortization method. Thus, *if the rate of interest on the sinking fun equals that on the loan, the sinking fund method of repayment is equivalent with the amortization method.*

7. Bonds

It is, of course, not common for an individual to take out a loan at a discount: the final payment is rarely more than the original loan. It is, however, common in the bond market. A coupon bond [1] is a type of security, typically sold by a company or a governmental unit, which in the simplest case, pays out a fixed sum (the *coupon*) at regular periods (typically quarterly or semi annually) for a predetermined number of years (the *term*). At the end of the term, the bond is said to have reached *maturity* at which time the holder receives an additional, predetermined payment, called the *redemption value* or *par value* of the bond. Thus, an \$8,000 par value, 5 year bond with \$200 quarterly coupons, would generate \$200 for 20 quarters, together with a final payment of \$8,000.

Bond are bought to yield a particular rate of return. *If we want a return rate of i, then the price of the bond is the present value of all of the expected payments at rate i.*

Bonds also carry a *face value*. Typically the face value equals the redemption value, although these values would differ if, say, the bond were redeemed prior to maturity. The bond is said to be "redeemed at par" if the redemption value equals the face value. Often the coupon is stated as a percentage of the face value F. *This*

[1] There are also *accumulation bonds* which are bonds where the interest on the bond is reinvested at the same rate as the principal and is returned to the customer only when the bond is redeemed. Hence, an accumulation bond is analyzable as a fixed sum invested at a fixed rate of compound interest for a fixed time period.

percentage is unrelated to the yield rate and is used only in determining the value of the coupon.

EXAMPLE 40. Determine the price of the \$8,000 bond mentioned above if it is purchased to yield 6%, compounded quarterly.

SOLUTION. Let $i = .06/4$. The present value of all of the payments is

$$200a_{\overline{20}|i} + (1+i)^{-20}8000 = 6469.55$$

which is the price of the bond. On the BA II Plus, we would first set $xP/Y = 4$. Then $N = 5$, $I/Y = 6$, $PMT = -200$, $FV = 8000$ and then press [CMP][PV].

Bond are in essence loans made by the purchaser to the seller. The purchase price is the amount of the loan. The yield rate is the interest rate of the loan and coupons are the payments on the loan. The redemption value is just a final payment. Thus, the following questions have the same answer:

Loan Question: How much could I borrow at 5% interest, compounded quarterly, if I am willing to pay \$150 twice a year for 7 years, together with a final payment of \$5000?

Bond Question: What is the purchase price of an 7 year par value \$5,000 bond with quarterly coupons of \$150 purchased to yield a 5% return, compounded quarterly?

Since bonds are loans, we may use the loan terminology and formulas to discuss bonds. Thus, we can, for example, discuss the balance of the bond, the principal adjustment, and whether the bond was bought at a premium or at a discount. Specifically, it will be at a discount if the redemption value is greater than or equal to the purchase price. We can also use amortization tables for bonds, with the columns appropriately relabeled. The exercises go into these points in greater detail.

The technique of using the present value of all of the future payments to compute the price of a bond might not always reflect the price we really should pay. Suppose, for example, that a 10 year, par value 10,000 bond with 8% semiannual coupon is purchased to yield 6% interest, compounded semiannually. Using the methods studied above, the price is computed to be 11,487.74.

Saying that the yield rate is 6% means that we should obtain the same total value of payments by investing 11,487.74 at 6% compounded semiannually. This sum, at 6% semiannual interest, yields

$$.03 \times 11,487.74 = 344.63$$

interest each 6 months. Note that 344.63 is less than the 400 semiannual coupon on the bond. The difference (55.37) is attributable to the fact that at the end of 10 years, we receive only the 10,000 redemption value of the bond and not our original 11,487.74 investment. Thus, we expect that payments of 55.37 at the end of every 6 months for 10 years, invested at 6% compounded semiannually, would accumulate to $10,000 - 11,487.74 = 1,487.74$–i.e.

$$55.37s_{\overline{20}|.03} = 1,487.74 \tag{36}$$

which is indeed correct. We think of the 344.63 portion of the payment as being the income from our original investment and the 55.37 portion as being used to replace the value lost from not getting the full investment returned at the end.

In the above description, we have in essence analyzed our bond as a loan being repaid using a sinking fund. To explain this, suppose that we borrow P at interest rate i per period for N periods which we repay by (a) paying the interest iP for each period, (b) making an additional final payment of F and (c) making regular payments of R into a sinking fund that earns interest at rate j. The payments R into the sinking fund must then accumulate to $P - F$–i.e.

$$Rs_{\overline{N}|j} = P - F \tag{37}$$

The total payment is then $R + iP$.

In the bond example, the final payment is the redemption value $F = 10,000$ and the loan amount is $P = 11,487.74$ so $P - F = 1,487.74$. The "iP" portion of the 400 coupon is 344.63 and the rest of the coupon (55.37) goes into the sinking fund. Equation (36) is then a special case of equation (37) where $i = j$. The validity of formula (36) follows from the theorem that if the interest rate on a loan equals the rate on the sinking fund than the sinking fund repayment method is equivalent with the amortization method.

There are situations where one analyzes bonds using sinking funds where the rate on the sinking fund differs from the yield rate on the bond. Suppose, for example, that the bond in question carries some risk. It is, perhaps, reasonable to consider that the funds being saved to replace the difference between the price of the bond and its redemption value are invested in securities which are more conservative than the bond and thus earn interest at a lower rate. In this case, the original price of the bond will be lower since these replacement funds will accumulate to a smaller amount. It is of course to be expected that a riskier bond will sell for less than a more secure bond, even if the coupons and desired yield rates are the same in each case.

Example 41. Find the price of a 10 year, par value 10,000 bond with 8% semiannual coupon purchased to yield 6% interest, compounded semiannually if the investor can replace the capital by a sinking fund earning 5% convertible semi-annually.

Solution. Let the price of the bond be P. The sinking fund must accumulate to $P - 10000$ to replace the capital–i.e. the semiannual payment into the sinking fund is

$$R = \frac{P - 10000}{s_{\overline{20}|.025}} = .03195(P - 10000)$$

The total semiannual payment is the coupon which, in this case, is 400. Hence

$$\begin{aligned} 400 &= R + iP \\ &= .03195(P - 10000) + .03P \\ &= .09195P - 691.47 \end{aligned}$$

Hence

$$P = \frac{400 + 601,47}{.09195} = 11,870.27$$

Exercises

1. What is the amount you can borrow on January 15, 2001 if you are willing to pay \$1250 on the 15th of each month, starting February 15, 2001 with

the final payment on July 15, 2013 if the interest rate is 6% compounded monthly?

2. What is the amount of each payment if you borrow \$18000 on a 60 month auto loan that is charging 10.8% (compounded monthly)?
3. What is the amount of each monthly payment on a 15 year mortgage that charges 8.4% (compounded monthly) if the purchaser needs to borrow \$149,500?
4. What is the amount of each monthly payment on a 30 year mortgage that charges 8.4% (compounded monthly) if the purchaser needs to borrow \$149,500? (Compare with previous problem.)
5. Bob Roarman will sell you a slightly used car for \$7,500 cash or you can buy the same car for 60 payments of only \$159 each (made at the end of each month). What rate of interest is Bob charging?

8. Continuous Processes

We have seen that for a given nominal yearly rate i, compounding monthly produces higher yields than yearly compounding. Compounding daily produces even higher yields. We can even compound every second. There is, however, a point at which the rate of compounding makes little difference. Specifically, if we compound n times a year at the nominal rate i, then P grows to

$$(1+\frac{i}{n})^n P$$

Using L'Hopital's rule (see Exercise ??) it can be shown that

$$\lim_{n\to\infty}(1+\frac{i}{n})^n P = e^i P$$

Hence *if we compound continuously at nominal rate i, the effective rate of return is j where*

$$1+j=e^i.$$

Thus, for example continuous compounding at nominal rate 3.3% produces an effective rate of

$$1+j=e^{.033}=1.033550539$$

for an effective rate of 3.355%. On the other hand, daily compounding yields

$$1+j=(1+\frac{.033}{360})^{360}=1.033549100$$

which produces more or less the same ROR.

It is also possible to imagine depositing money continuously. Suppose that our company earns C dollars per year, spread out evenly throughout the year. At the end of each day, we deposit the day's income into an account that earns interest at an annual effective rate of i. Thus, each deposit is $C/360$ dollars. (Recall that by our convention, a year has 360 days.) The daily interest rate is defined by

$$1+j=(1+i)^{1/360}.$$

Hence, after n years, our amount function is

$$\begin{aligned} A(n) &= \frac{(1+j)^{360n}-1}{j}\frac{C}{360} \\ &= \frac{(1+i)^{n}-1}{j}\frac{C}{360} \\ &= is_{\overline{n}|i}\frac{1}{360j}C \\ &= is_{\overline{n}|i}\frac{1/360}{(1+i)^{1/360}-1}C \end{aligned}$$

We could, in theory, deposit our income at the end of every second, in which $A(n)$ would be given by the same formula, except that 360 would be replaced by the number of seconds in a year. In general, if we deposit our profits m times a year, our amount is

$$A_m(n) = is_{\overline{n}|i}\frac{1/m}{(1+i)^{1/m}-1}C \tag{38}$$

As m tends to ∞, $1/m$ tends to zero. From L'Hopital's rule

$$\lim_{x\to 0}\frac{x}{(1+i)^x-1} = \frac{1}{\ln(1+i)} = \frac{1}{\delta}$$

where δ is the force of interest.

Hence, taking the limit as m tends to infinity in formula 38 motivates the following definition.

Definition 10. *Depositing funds continuously at C per compounding period into an account that yields interest rate i per compounding period accumulates to*

$$A(n) = \overline{s}_{\overline{n}|i}C$$

after n compounding periods where

$$\overline{s}_{\overline{s}|i} = \frac{i}{\delta}s_{\overline{n}|i} \tag{39}$$

and where $\delta = \ln(1+i)$ is the force of interest.

Remark. We do not require that n be an integer in formula 39.

Example 42. If you invest \$ P at the end of each month for 18 years in an account that earns 7.1% interest, compounded monthly, your sums will accumulate to \$250,000. How much would you have accumulated if you had been depositing continuously at the rate of \$ P per month?

Solution. We adopt the month as our basic unit of time. Let $i = .071/12$. The given tells us that

$$s_{\overline{240}|i}P = 250000$$

Hence, the answer is

$$\begin{aligned} \overline{s}_{\overline{240}|i}P &= \frac{i}{\delta}s_{\overline{240}|i}P \\ &= \frac{i}{\delta}250000 = (1.003038647)250000 = 250759.6618 \end{aligned}$$

9. Depreciation Methods

When a company buys an asset, say a computer, the value of the asset of course declines over time. This loss of value can at times be considered as a business expense; hence as a tax deduction. Or if a company is to be sold, the value of all of its assets must be determined. In both cases it becomes necessary to compute the loss in value (the *depreciation*) of the asset.

There are many ways of computing depreciation which may yield very different answers. The choice of a particular depreciation technique is often dependent on its intended use. For tax purposes, one might, for example, want to choose a technique that depreciated the equipment as quickly as possible to get the write off as soon as possible. When selling a company, one might want to choose a technique that depreciated the equipment more slowly in order to create a larger valuation of assets.

All of the depreciation techniques we discuss begin with an initial value A of the asset, a time interval $[0, n]$ over which the depreciation occurs, and a final value S, the *salvage value*, of the asset. For convenience, we will think of n as representing years although it could, of course, represent any of the other common units of time e.g. days, months, quarters, *etc.* The goal is to assign a value $B(t)$, the *book value*, to the asset for all t in the given time interval. One is also interested in the *depreciation charge* $D(t)$, which is the amount the asset depreciated over the year ending at time t. Mathematically

$$D(t) = B(t-1) - B(t).$$

We think of $D(t)$ as representing that part of the cost of using the asset over the year which is attributable to depreciation.

The simplest depreciation technique is the *straight line* method under which the book value is assumed to decrease linearly from A to S over the stated time interval. Thus

$$B(t) = A - \frac{A-S}{n}t \tag{40}$$

In this case, the depreciation charges are constant since

$$\begin{aligned} D(t) &= B(t-1) - B(t) \\ &= A - \frac{A-S}{n}(t-1) - (A - \frac{A-S}{n}t) \\ &= \frac{A-S}{n} \end{aligned}$$

Example 43. A laptop computer bought in Jan. 2000 for \$4500 is depreciated over a 5 year period assuming the straight line method and a salvage value of \$1000. What is its book value in Jan. 2003? What was the depreciation charge for 2003?

Solution. From equation (40)

$$\begin{aligned} B(t) &= 4500 - \frac{4500-1000}{5}t + \\ &= 4500 - 700t \end{aligned}$$

Thus, the book value on Jan. 1, 2003 is

$$B(3) = 4500 - 2100 = 2400$$

The depreciation charge for 2003 is $B(2) - B(3) = 700$.

After the straight line method, the next simplest depreciation technique is the *constant percentage* (*compound discount*, *declining balance*) method. Under this method, the value of the asset is assumed to decline by a fixed fraction d each year, reaching its salvage value after n years. Mathematically, this is equivalent with investing A into an account that earns compound interest at rate $i = -d$ where $d > 0$. Thus, under this method

$$B(t) = (1-d)^t A$$

where d is determined by the equation

$$S = (1-d)^n A$$

The number d is referred to as the *the rate of discount* for the asset.

EXAMPLE 44. Redo Example 43 using the declining balance method.

SOLUTION. The rate of discount is determined by the equation

$$(1-d)^5 4500 = 1000$$

which implies

$$1 - d = (\frac{1}{4.5})^{\frac{1}{5}} = .74$$

Thus, the book value on Jan. 1, 2003 is

$$B(3) = (.74)^3 4500 = 1825.09$$

The depreciation charge for 2003 is

$$B(2) - B(3) = 640.53.$$

A somewhat more natural depreciation method is the *sinking fund* method. The idea is that we imagine investing a fixed sum of money into an account (called the *sinking fund*) at the end of each year that will be used to replace the asset at the end of the depreciation period. The sinking fund must accumulate to $A - S$ at the end of the depreciation period since we will, presumably, get the salvage value S when we sell the asset. Hence, our annual payment into the sinking fund will be

$$C = \frac{A-S}{s_{\overline{n}|i}} \tag{41}$$

where n is the number of years over which the asset is depreciated and i is the assumed interest rate. The book value of the asset at the end of a given year is the difference between A and the current balance of the sinking fund. Hence

$$B(k) = A - Cs_{\overline{k}|i}.$$

EXAMPLE 45. Redo Example 43 using the sinking fund method. Assume that the sinking fund earns 6% interest per year.

SOLUTION. The sinking fund must accumulate to $A - S = 3500$ in five years at 6% interest per year. Hence, the monthly payments are

$$P = 3500/s_{\overline{20}|.06}$$

which compute on the BAII by setting FV=−3500, PV=0, P/Y=1, I/I=.06, N=5, an CPT PMT, getting PMT=585.74. The book value $B(3)$ is

$$B(3) = 4500 - 585.74s_{\overline{3}|.06}$$

To compute this on the BAII, we simply change the value of N to 3 and ask the BAII to compute FV getting -1976.66 which we add to 4500, finding $B(3) = 2523.34$.

The depreciation charge for 2003 is

$$B(2) - B(3) = 697.63.$$

As mentioned above, under the sinking fund method, the book value at time t is

$$B(t) = A - B_s(t)$$

where $B_s(t)$ is the amount in the sinking fund at time t. The depreciation charge is then

$$D(t) = B_s(t) - B_s(t-1).$$

Hence, *the depreciation charge is exactly the principal adjustment for the sinking fund.* Thus, from formula (34)

$$D(t) = (1+i)^{t-1}|i(A-S) - C|.$$

It follows that the depreciation charge increases exponentially as t increases. Hence, the greatest depreciation occurs in the later years.

Often, it is possible to take depreciation as a tax write-off. In such cases, it might be desirable to have the largest depreciation occur at the beginning, on the principle that it is advantageous to get the write-off sooner rather than later. Our final depreciation method accomplishes this goal by assuming the depreciation charge $D(t)$ *decreases* by a fixed amount C each year–i.e.

$$D(t+1) = D(t) - C \tag{42}$$

We also assume that

$$C = D(n)$$

so that according to formula (42), the depreciation charge in the $(n+1)$st year is 0. The unique function satisfying these two conditions is

$$D(t) = (n+1-t)C. \tag{43}$$

The book value $B(m)$ in year m is the original value of the asset minus the total of the depreciation charges to date. Hence

$$B(m) = A - (D(1) + D(2) + \cdots + D(m)) \tag{44}$$

Since $B(n) = S$, we see that

$$\begin{aligned} A - S &= D(1) + D(2) + \cdots + D(n) \\ &= (n + (n-1) + \cdots + 1)C \\ &= S_n C \end{aligned} \tag{45}$$

where

$$S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{46}$$

Hence

$$C = \frac{A-S}{S_n} \tag{47}$$

It follows from the first formula in (45), together with equation (43), that

$$\begin{aligned} A - (D(1) + D(2) + \cdots + D(m)) &= S + D(m+1) + D(m+2) + \cdots + D(n) \\ &= S + (n - m + (n - m - 1) + \cdots + 1)C \\ &= S + S_{n-m}C \end{aligned}$$

where S_m is as in equation (46). Hence, from formulas (44) and 47),

$$B(m) = S + \frac{S_{n-m}}{S_n}(A - S) \tag{48}$$

This depreciation method is called the *Sum of the digits* method because S_m is the sum of the digits from 1 to m. It is also called the "method of 78" because, from formula (46), the sum of the digits from 1 to 12 is 78.

Example 46. Redo Example 43 using the sum of the digits method.

Solution. From formulas (48) and (46)

$$\begin{aligned} B(m) &= 1000 + \frac{S_{5-m}}{S_5}(4500 - 1000) \\ &= 1000 + \frac{(5-m)(5-m+1)}{2 \cdot 15} 3500 \end{aligned}$$

Thus,

$$B(3) = 1700.$$

The depreciation charge in year 3 is

$$D(3) = B(2) - B(3) = 2400 - 1700 = 700$$

10. Capitalization Costs

Imagine that you are managing a factory that produces squidgets. You would like to determine the cost of operating each squidget machine for a year. We will assume that the cost does not change from year to year. Thus we are ignoring factors such as inflation and changes in the machine's productivity as it ages.

Suppose that the original purchase price for the asset was A. In the first year of operation, we lose the interest on A which is iA where we assume that A was invested at rate i per year. The asset also depreciates. Using the sinking fund method of depreciation, the depreciation expense is the payment into the sinking fund which is given by formula (41). Finally, there are fixed costs such as labor, utilities, maintenance, etc. Hence, the total cost C of running the machine for one year is

$$C = Ai + \frac{S - A}{s_{\overline{n}|j}} + M \tag{49}$$

where i is the interest rate on the funds used to purchase the asset, n is the number of years over which the asset is depreciated, j is the interest rate on the sinking fund use to replace the asset, S is the salvage value of the asset and M is the fixed cost of operation.

It is natural to assume that $i = j$ in which case this formula simplifies as follows

$$
\begin{aligned}
C &= Ai + \frac{A - S}{s_{\overline{n}|i}} + M \\
&= \frac{Ais_{\overline{n}|i} + A}{s_{\overline{n}|i}} - \frac{S}{s_{\overline{n}|i}} + M \\
&= \frac{A((1+i)^n - 1) + A}{s_{\overline{n}|i}} - \frac{S}{s_{\overline{n}|i}} + M \qquad (50) \\
&= \frac{A(1+i)^n}{s_{\overline{n}|i}} - \frac{S}{s_{\overline{n}|i}} + M \\
&= \frac{A}{a_{\overline{n}|i}} - \frac{S}{s_{\overline{n}|i}} + M
\end{aligned}
$$

This equation has a simple interpretation. It says that

$$C = P - Q + M$$

where

$$
\begin{aligned}
A &= Pa_{\overline{n}|i} \\
S &= Qs_{\overline{n}|i}
\end{aligned}
$$

The payments P represent the present value of the asset A spread equally over the life of the asset. Similarly, the payments Q represent the future salvage value of the asset, spread out over the life of the asset. $P - Q$ is then the difference between one years share of the original cost of the asset and one year's share of the replacement value of the asset.

Operating costs are often used to compare the price of two competing assets being considered for purchase. In this case, one typically compares the cost per unit output.

Example 47. For \$2,500 we can buy a computer which will last for 5 years and has a salvage value of \$1,000. For \$3,500 we can buy a computer that is 20% faster, lasts for 6 years and has the same salvage value. Both computers have the same maintenance expense. At a 7% interest rate, which computer represents the better value?

Solution. Since both computers have the same maintenance expense M, we may assume that the annual maintenance expense is 0. Hence, from formula (50), the annual cost of the kth computer is C_k where

$$
\begin{aligned}
C_1 &= \frac{2500}{a_{\overline{5}|.07}} - \frac{1000}{s_{\overline{5}|.07}} \\
&= 609.73 - 173.89 = 435.84 \\
C_2 &= \frac{3500}{a_{\overline{6}|.07}} - \frac{1000}{s_{\overline{6}|.07}} \\
&= 734.29 - 139.80 = 594.49
\end{aligned}
$$

However, the second computer is 20% faster. Hence, its annual cost per unit output is $594.49/1.2 = 495.41$. The first computer is still the better deal.

Figure 2. Time Line for Example 6

\$1000 −\$500 \$1500 ?

FIGURE 3. Time Line for Example 7

550 −100 −300 50 ?

Figure 4. Time Line for Example 9

100k 100k 100k 100k

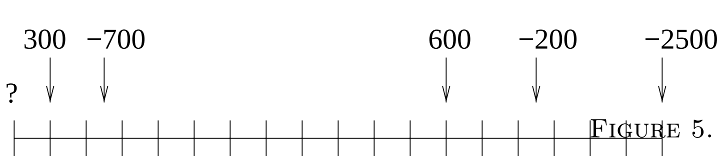

Figure 5. Time Line for Example 11

FIGURE 6. Time Line for Example 15

50 −115 66