Joint Parameter Estimation of the Ornstein-Uhlenbeck SDE driven by Fractional Brownian Motion

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Main Objective: To study the GMM (Generalized Method of Moments) joint estimator of the drift and memory parameters of the Ornstein-Uhlenbeck SDE driven by fBm.

Joint work with Prof. Frederi Viens (Department of Statistics, Purdue University).
Outline

1. Preliminaries
2. Joint estimation of Gaussian stationary processes
3. fOU Case
4. Simulation
1 Preliminaries

2 Joint estimation of Gaussian stationary processes

3 fOU Case

4 Simulation
The fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is the centered Gaussian process $B^H_t$, continuous a.s. with:

$$\text{Cov}(B^H_t, B^H_s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \quad t, s \in \mathbb{R}.$$ 

Some properties:

- **Self-Similarity:** for each $a > 0$ $B^H_{at} \overset{d}{=} a^H B^H_t$. 
- It admits an integral representation with respect to the standard Brownian motion over a finite interval:

$$B^H_t = \int_0^t K_H(t, s)dW(s).$$
Fractional Gaussian noise (fGn)

- Definition:

\[ N_t^H := B_t^H - B_{t-\alpha}^H \]

where \( \alpha > 0 \).

- Gaussian and stationary process.

- Long-memory behavior of the increments when \( H > \frac{1}{2} \) (in the sense that \( \sum_n \rho(n) = \infty \)).

- Ergodicity.
Fractional Gaussian noise (fGn)

- Autocovariance function:
  \[
  \rho_\theta(t) = \frac{1}{2} \left[ |t + \alpha|^{2H} + |t - \alpha|^{2H} - 2|t|^{2H} \right]
  \]

- Spectral density:
  \[
  f_\theta(t) = 2c_H(1 - \cos t)|t|^{-1-2H}
  \]

where \( c_H = \frac{\sin(\pi H)\Gamma(2H+1)}{2\pi} \). (Beran, 1994).
Estimation of $H$ (fBm and fGn)

Classical methods:

- $R/S$ statistic (Hurst 1951), Variance plot (Heuristic method).

MLE Methods:

- Whittle’s estimator.
Methods using variations (filters):

- Coeurjolly (2001): consistent estimator of $H \in (0, 1)$ based on the asymptotic behavior of discrete variations of the fBm. Asymptotic normality for $H < 3/4$.

Cheridito (2003) Take $\lambda, \sigma > 0$ and $\zeta$ a.s bounded r.v. The Langevin equation:

$$X_t = \zeta - \lambda \int_0^t X_s ds + \sigma B_t^H, \quad t \geq 0$$

has as an unique strong solution and it is called the fractional Ornstein-Uhlenbeck process:

$$\zeta X_t = e^{-\lambda t} \left( \zeta + \sigma \int_0^t e^{\lambda u} dB_u^H \right), \quad t \leq T$$

and this integral exists in the Riemann-Stieltjes sense.
Cheridito (2003):
- Stationary solution (fOU process):
  \[ X_t = \sigma \int_{-\infty}^{t} e^{-\lambda(t-u)} dB_u^H, \quad t > 0 \]
- Autocovariance function (Pipiras and Taqqu, 2000):
  \[
  \rho_\theta(t) = 2\sigma^2 c_H \int_0^\infty \cos(tx) \frac{x^{1-2H}}{\lambda^2 + x^2} dx
  \]
  where \( c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \).
Cheridito (2003):

- \( X_t \) has long memory when \( H > \frac{1}{2} \), due to the following approximation when \( x \) is large:

\[
\rho_\theta(x) = \frac{H(2H - 1)}{\lambda} x^{2H-2} + O(x^{2H-4}).
\]

- \( X_t \) is ergodic.

- \( X_t \) is not self-similar, but it exhibits asymptotic selfsimilarity (Bonami and Estrade, 2003):

\[
f_\theta(x) = c_H |x|^{-1-2H} + O(|x|^{-3-2H}).
\]
Estimation of $\lambda$ given $H$ (OU-fBm)

MLE estimators:
- Kleptsyna and Le Breton (2002):
  - MLE estimator based on Girsanov formula for fBm.
  - Strong consistency when $H > 1/2$.
- Tudor and Viens (2006):
  - Extended the K&L result to more general drift conditions.
  - Strong consistency of MLE estimator when $H < \frac{1}{2}$ using Malliavin calculus.

They work with the non-stationary case.
Estimation of $\lambda$ given $H$ (Least-Squares methods)

Hu and Nualart (2010)

- Estimate of $\lambda$ which is strongly consistent for $H \geq \frac{1}{2}$.

$$\tilde{\lambda}_T = \left( \frac{1}{H\Gamma(2H)T} \int_0^T X_t^2 dt \right)^{-\frac{1}{2H}}$$

- $\tilde{\lambda}_T$ is asymptotically normal if $H \in \left( \frac{1}{2}, \frac{3}{4} \right)$

- The proofs of these results rely mostly on Malliavin calculus techniques.
Methods based on variations:

- Biermé et al (2011): For fixed $T$, they use the results in Biermé and Richard (2006) to prove consistency and asymptotic normality of the joint estimator of $(H, \sigma^2)$ for any stationary gaussian process with asymptotic self-similarity (Infill-asymptotics case).

- Brouste and Iacus (2012): For $T \to \infty$ and $\alpha \to 0$ they proved consistency and asymptotic normality when $\frac{1}{2} < H < \frac{3}{4}$ for the pair $(H, \sigma^2)$ (non-stationary case).
Outline

1. Preliminaries

2. Joint estimation of Gaussian stationary processes

3. fOU Case

4. Simulation
Preliminaries

- \( X_t \): real-valued centered gaussian stationary process with spectral density \( f_{\theta_0}(x) \).
- \( f_{\theta}(x) \): continuous function with respect to \( x \), continuously differentiable with respect to \( \theta \).
- \( \theta \) belongs a compact set \( \Theta \subset \mathbb{R}^p \).
- Bochner’s theorem:

\[
\rho_{\theta}(s) := \text{Cov}(X_{t+s}, X_t) = \int_{\mathbb{R}} \cos(sx)f_{\theta}(x)dx
\]
If $\rho_\theta(s)$ is a continuous function of $s$, then the process $X_t$ is ergodic.

**Assumption 1**

*Take $\alpha > 0$ and $L$ a positive integer. Then there exists $k \in \{0, 1, \ldots, L\}$ such that $\rho_\theta(\alpha k)$ is an injective function of $\theta$.***
Recall: \( a(l) := (a_0(l), \ldots, a_L(l)) \) is a discrete filter of length \( L + 1 \) and order \( l \) for \( L \in \mathbb{Z}^+ \) and \( l \in \{0, \ldots, L\} \) if

\[
\sum_{k=0}^{L} a_k(l)k^p = 0 \quad \text{for } 0 \leq p \leq l - 1
\]

\[
\sum_{k=0}^{L} a_k(l)k^p \neq 0 \quad \text{if } p = l.
\]

Examples: finite-difference filters, Daubechies filters (wavelets).

Assume that we can choose \( L \) filters with orders \( l_i \in \{1, \ldots, L\} \) for \( i = 1, \ldots, L \) and a extra filter with \( l_0 = 0 \).
Define the filtered process of order $l_i$ and step size $\Delta_f > 0$ at $t \geq 0$ as:

$$\varphi_i(X_t) := \sum_{q=0}^{L} a_q(l_i)X_{t-\Delta_f q}$$

Its expected value is: $V_i(\theta_0) := E[\varphi_i(X_t)^2] = \sum_{k=0}^{L} b_k(l_i)\rho_{\theta_0}(\Delta_f k)$.

Define the set of moment equations by:

$$g(X_t, \theta) := (g_0(X_t, \theta), \ldots, g_L(X_t, \theta))'$$

where

$$g_i(X_t, \theta) = \varphi_i(X_t)^2 - V_i(\theta), \quad \text{for } 0 \leq i \leq L.$$
Assume that we have observed the stationary process $X_t$ at times $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ and $\alpha := t_i - t_{i-1} > 0$ (fixed).

Assume there exists a sequence of symmetric positive-definite random matrices $\{\hat{A}_N\}$ such that $\hat{A}_N \xrightarrow{P} A$, and $A > 0$.

Define:

- $\hat{g}_N(\theta) := \frac{1}{N-L+1} \sum_{i=L}^{N} g(X_{t_i}, \theta)$. (sample moments)

- $\hat{Q}_N(\theta) := \hat{g}_N(\theta)' \hat{A}_N \hat{g}_N(\theta)$.

- $Q_0(\theta) := E[g(X_t, \theta)]' AE[g(X_t, \theta)]$. 
Define the GMM estimator of $\theta_0$:

$$\hat{\theta}_N := \arg\min_{\theta \in \Theta} \hat{Q}_N(\theta)$$

$$= \arg\min_{\theta \in \Theta} \left( \frac{1}{N} \sum_{i=1}^{N} g(X_{t_i}, \theta) \right)^T \hat{A}_N \left( \frac{1}{N} \sum_{i=1}^{N} g(X_{t_i}, \theta) \right).$$
Lemma 1

Under the above assumptions:

(i) \[
\sup_{\theta \in \Theta} |\hat{Q}_N(\theta) - Q_0(\theta)| \xrightarrow{a.s.} 0.
\]

(ii) \[Q_0(\theta) = 0.\]

if and only if \(\theta = \theta_0\).
Consistency

Key aspects of proof:

- Ergodicity of $X_t$
- Continuity of $\rho_\theta(\cdot)$ over the compact set $\Theta$.
- Injectivity of:

$$\rho_\theta(\alpha) := \begin{bmatrix} \rho_\theta(\alpha \cdot 0) \\ \vdots \\ \rho_\theta(\alpha \cdot L) \end{bmatrix}$$

- Results of Newey and McFadden, 1994. (GMM)
We have all the conditions to apply:

**Theorem 1 (Newey and McFadden, 1994)**

*Under the above assumptions, it holds that:*

$$\hat{\theta}_N \xrightarrow{a.s.} \theta_0.$$
Asymptotic Normality of the sample moments

Denote:

\[ \hat{G}_N(\theta) := \nabla_\theta \hat{g}_N(\theta), \quad G(\theta) := E[\nabla_\theta g(X_t, \theta)] \]

Assumption 2
For fixed \( \alpha > 0 \), assume that the \((L+1) \times p\) matrix \( \nabla_\theta \rho_\theta(\alpha) \) is a full-column rank matrix for any \( \theta \in \Theta \).

We can prove that: \( \hat{G}_N(\theta) = G(\theta) = -\nabla_\theta V(\theta) \), where \( V(\theta) = (V_0(\theta), \ldots, V_L(\theta)) \).
And based on Biermé et al (2011) article, let us assume:

**Assumption 3 (Biermé et al’s condition)**

Assume that for any \( l, l' \in \{l_0, \ldots, l_L\} \) we have that:

\[
R(u|l, l') := \min \left( 1, |u|^{l+l'} \right) \sum_{p \in \mathbb{Z}} f_{\theta_0} \left( \frac{u + 2\pi p}{\alpha} \right) \in L^2((−\pi, \pi))
\]
Lemma 2

Under Assumptions 1-3

\[ \sqrt{N} \hat{g}_N(\theta_0) \xrightarrow{d} N(0, \Omega) \]

where

\[ \Omega_{ij} = 2\alpha^{-2} \int_{-\pi}^{\pi} |P_{a_{i_1}}[\cos u]|^2 \cdot |P_{a_{i_j}}[\cos u]|^2 \bar{f}_{\theta_0}(u/\alpha)^2 \, du \]

and \( \bar{f}_{\theta_0}(u) := \sum_{p \in \mathbb{Z}} f_{\theta_0}(u + 2\pi p) \).

Note: \( P_{a_l}(x) := \sum_{k=0}^{L} a_k(l)x^k \).
Sketch of proof

Let $V_D(\theta_0) := \text{Diag} \left( \frac{1}{V_j(\theta_0)} \right)_{j \in \{0, \ldots, L\}}$. We scale the vector $g(X_t, \theta_0)$ as follows:

$$\sqrt{N} V_D(\theta_0) \hat{g}_N(\theta_0) = \frac{\sqrt{N}}{N - L + 1} \sum_{i=L}^{N} (H_2(Z_{l_j, t_i}))_{j \in \{0, \ldots, L\}}$$

where $Z_{l_j, t_i} := \frac{\varphi_j(X_{t_i})}{\sqrt{V_j(\theta_0)}}$ and $H_2(\cdot)$ is the 2nd-order Hermite process.

Use the vector-valued version of the Breuer-Major theorem with spectral-information conditions (Biermé et al, 2011) to deduce the asymptotic behavior of the previous sum.
Asymptotic behavior of the error

Let $\varepsilon_N := \hat{\theta}_N - \theta_0$. Using the mean value theorem:

$$
\varepsilon_N := \hat{\theta}_N - \theta_0 = -\psi_N(\bar{\theta}_N, \hat{\theta}_N) \cdot \hat{g}_N(\theta_0)
$$

where $\psi_N(\bar{\theta}_N, \hat{\theta}_N) := [G(\hat{\theta}_N)' \hat{A}_N G(\bar{\theta}_N)]^{-1} \cdot G(\hat{\theta}_N)' \hat{A}_N$.

Also note that:

$$
\psi_N(\bar{\theta}_N, \hat{\theta}_N) \xrightarrow{p} [G(\theta)' A G(\theta)]^{-1} G(\theta)' A
$$

then we can bound $E[\|\psi_N(\bar{\theta}_N, \hat{\theta}_N)\|^{4p}]$ for any $p > 0$. 

Asymptotic behavior of the error

- $X_t$ can be represented as a Wiener-Ito integral with respect to the standard Brownian motion:

  \[ X_{t_k} = I_1(A_k(\cdot|\theta_0)) \]

  where $A_k(x|\theta_0) := \cos(\alpha k x) \tilde{f}_{\theta_0}(x)$.

- Using the multiplication rule of Wiener integrals:

  \[ (\hat{g}_N(\theta))_i = l_2[B_{i,j}(\cdot|\theta)] \]

  where $B_{i,j}(\cdot|\theta)$ is a kernel depending on the filter $\mathbf{a}$. 
Asymptotic behavior of the error

- We already proved a CLT for $\hat{g}_N(\theta_0)$, hence, for all $i$:
  \[
  E[|\hat{g}_N(\theta_0)|^2] = O(N^{-1})
  \]

- Let $p > 0$, we can use the equivalence of $L^p$-norms of a fixed Weiner chaos to get:
  \[
  E[\|\hat{g}_N(\theta_0)\|^{4p}] < \frac{\tau_{p,L}}{N^{2p}}
  \]

- By Borel-Cantelli, we conclude:
Corollary 1

Under Assumptions 1-3 we have:

(i) \( E[\|\hat{\theta}_N - \theta_0\|^2] = O(N^{-1}) \)

(ii) \( N^\gamma \|\hat{\theta}_N - \theta_0\| \xrightarrow{a.s.} 0 \) for any \( \gamma < \frac{1}{2} \).
Asymptotic behavior of the error

And we can generalize the previous result as follows:

**Theorem 2**

Let $\hat{\theta}_N$ the GMM estimator of $\theta_0$. Assume there exists a diagonal $(L + 1) \times (L + 1)$ matrix $D_N(\theta_0)$ such that:

$$E[D_N^{(i,i)}(\theta_0)|\hat{g}_N(\theta_0)_i|^2] = O(1)$$

for all $i \in \{0, \ldots, L\}$. Then:

(i) $E[\|\hat{\theta}_N - \theta_0\|^2] = O\left(\max_{0 \leq i \leq L}\{D_N^{(i,i)}(\theta_0)\}\right)$

(ii) If $\max_{0 \leq i \leq L}\{D_N^{(i,i)}(\theta_0)\} = f(N)/N^\nu$, for $f(N) = o(N)$ and $\nu > 0$ then for any $\gamma < \nu/2$:

$$N^\gamma \|\varepsilon_N\| \xrightarrow{a.s.} 0.$$
Theorem 3

Let $X_t$ a Gaussian stationary process with parameter $\theta_0$. If $\hat{\theta}_N$ is the GMM estimator of $\theta_0$, then under Assumptions 1-3, it holds that:

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \overset{d}{\rightarrow} N(0, C(\theta_0)\Omega C(\theta_0)')$$

where $C(\theta_0) = [G(\theta_0)'AG(\theta_0)]^{-1} G(\theta_0)'A$.

Key ideas of the proof (Newey and McFadden, 1994; Hansen, 1982):

- Linear behavior of $\varepsilon_N = \hat{\theta}_N - \theta_0$.
- Slutsky’s theorem.
Efficiency

- The GMM asymptotic variance is minimized by taking $A = \Omega^{-1}$.
- There are numerical techniques to compute $\hat{A}_N$ such that $\hat{A}_N \xrightarrow{p} \Omega^{-1}$, for example (Hansen et al., 1996):
  - Two-step estimators.
  - Iterative estimator.
  - Continuous-updating estimator.
Outline

1 Preliminaries

2 Joint estimation of Gaussian stationary processes

3 fOU Case

4 Simulation
Assume that there exist a closed rectangle $\Theta \subset \mathbb{R}^2$ such that $\theta = (H, \lambda) \in \Theta$, and assume that $\sigma = 1$.

- $\rho_\theta(t)$ and its partial derivatives are continuous functions of $\theta$.
- The Assumptions 1 and 2 are difficult to confirm due to the analytical complexity of the covariance function $\rho_\theta(t)$. we decided to check these two assumptions at least locally, by checking numerically that:

$$\det \begin{bmatrix}
\frac{\partial \rho_\theta(0)}{\partial H} & \frac{\partial \rho_\theta(0)}{\partial \lambda} \\
\frac{\partial \rho_\theta(\alpha)}{\partial H} & \frac{\partial \rho_\theta(\alpha)}{\partial \lambda}
\end{bmatrix} \neq 0$$

for different values of $\theta$ and $\alpha$.

- The injectivity approximately holds for $0 < \lambda < 5$ and $H > 0.3$. 


Lemma 3

Let $X_t$ be the stationary fOU process with parameters $\theta = (H, \lambda)$. The Assumption 3 (Biermé et al’s condition) holds under the following two cases:

- **Case 1** If $l + l' > 1$ then it holds for all $H \in (0, 1)$.
- **Case 2** If $l + l' \leq 1$ then it holds if $H \in (0, \frac{3}{4})$.

Conclusion: Assumption 3 is not valid for the first component of $\hat{g}_N$ if $H \geq \frac{3}{4}$. 

Asymptotic Normality of the sample errors

Using the previous lemma, for $H < \frac{3}{4}$:

$$\sqrt{N} \mathbf{g}_N(\theta) \xrightarrow{d} N(0, \Lambda(\theta))$$

where the covariance matrix $\Lambda(\theta)$ has entries:

$$\Lambda_{ij}(\theta) = 2c_H^2 \alpha^{4H} \int_{-\pi}^{\pi} \left| P_{a_i} [\cos u] \right|^2 \left| P_{a_j} [\cos u] \right|^2 \sum_{p \in \mathbb{Z}} \frac{|u + 2\pi p|^{1-2H}}{(u + 2\pi p)^2 + (\lambda \alpha)^2} \right)^2 \, du$$

$$\mathcal{K}_{i,j}(\alpha)$$
Lemma 4

Assume that \( X_t \) is a stationary fOU process with parameters \( \theta = (H, \lambda) \) where \( H \geq \frac{3}{4} \). Then:

(i) If \( H = \frac{3}{4} \):

\[
\sqrt{\frac{N}{\log N}} \left( \hat{g}_N(\theta) \right)_0 \xrightarrow{d} N(0, 2\alpha^{-1} c_\theta^2)
\]

where \( c_\theta = \frac{H(2H-1)}{\lambda} = \frac{3}{8\lambda} \).

(ii) If \( H > \frac{3}{4} \), \( (\hat{g}_N(\theta))_0 \) does not converge to a normal law, or even a second-chaos law. However,

\[
E \left[ \left| N^{2-2H}(\hat{g}_N(\theta))_0 \right|^2 \right] = O(1).
\]
Sketch of proof

- Denote: \( \hat{g}_{N,0}(\theta) = (\hat{g}_N(\theta))_0 \).
- For \( H = \frac{3}{4} \), as \( N \to \infty \):
  \[
  E \left[ \frac{N}{\log N} |\hat{g}_{N,0}(\theta)|^2 \right] \to 2\alpha^{-1} c^2_{\theta} := \tilde{c}_1
  \]
  where \( c_{\theta} := \frac{H(2H-1)}{\lambda} \).
- For \( H > \frac{3}{4} \):
  \[
  E[|N^{2-2H}\hat{g}_{N,0}(\theta)|^2] \to \frac{2\alpha^{4H-4} c^2_{\theta}}{(2H-1)(4H-3)} := \tilde{c}_2
  \]
Sketch of proof

If \( F_N = \begin{cases} \sqrt{\frac{N}{\tilde{c}_1 \log N}} \hat{g}_{N,0}(\theta) & \text{if } H = \frac{3}{4} \\ \sqrt{\frac{N^{4-4H}}{\tilde{c}_2}} \hat{g}_{N,0}(\theta) & \text{if } H > \frac{3}{4} \end{cases} \) then we need to prove

\[ \| DF_N \|_{L^2(\Omega)}^2 \xrightarrow{\mathcal{H}} 2 \]

where \( \mathcal{H} = L^2((-\pi, \pi)) \).

This result is valid only if \( H = \frac{3}{4} \).

Hence we can use Nualart and Ortiz-Latorre (2008) to conclude asymptotic normality of \( F_N \).
Sketch of proof

- In the case $H > \frac{3}{4}$ we can use the same criteria to conclude that $F_N$ does not have a normal limit in law.

- Furthermore, $F_N$ has this kernel:

\[
I_N(r, s) := \frac{N^{1-2H}}{\sqrt{\tilde{c}_2}} \sum_{j=1}^{N} (A_j \otimes A_j)(r, s)
\]

- It can be proved that $I_N(r, s)$ is not a Cauchy sequence in $\mathcal{H}^2$. Then $F_N$ does not have a 2nd-chaos limit.
Asymptotic behavior of the error

Proposition 1

Let $X_t$ be a fOU process with parameters $\theta = (H, \lambda)$. Then the GMM estimate $\hat{\theta}_N$ satisfies:

(i) $E[\|\hat{\theta}_N - \theta\|^2] = \begin{cases} O(N^{-1}) & \text{if } H \in (0, \frac{3}{4}) \\ O\left(\frac{\log N}{N}\right) & \text{if } H = \frac{3}{4} \\ O(N^{4H-4}) & \text{if } H \in (\frac{3}{4}, 1) \end{cases}$

(ii) As $N$ goes to infinity:

$$N^\gamma \|\hat{\theta}_N - \theta\| \xrightarrow{a.s.} 0$$

for all $\gamma < \frac{1}{2}$ (if $H \leq \frac{3}{4}$). Otherwise for all $\gamma < 2 - 2H$, if $H > \frac{3}{4}$. 
Asymptotics of $\hat{\theta}_N$:

**Proposition 2**

Let $X_t$ be the stationary fOU process with parameters $\theta = (H, \lambda)$. Then, for any positive-definite matrix $A$, the GMM estimator of $\theta$ is consistent for any $H \in (0, 1)$ and:

- If $H \in (0, \frac{3}{4})$:
  \[
  \sqrt{N}(\hat{\theta}_N - \theta) \xrightarrow{d} N(0, C(\theta)\Lambda C(\theta)')
  \]
  where $C(\theta) = [G(\theta)'AG(\theta)]^{-1}G(\theta)'A$ and $\Lambda = 2c_H^2\alpha^4HK(\alpha)$.

- If $H = \frac{3}{4}$, then
  \[
  \sqrt{\frac{N}{\log N}}(\hat{\theta}_N - \theta) \xrightarrow{d} N \left( 0, \frac{2c_\theta^2}{\alpha}(C(\theta))_1'(C(\theta))_1 \right)
  \]
  where $(C(\theta))_1$ is the first column of $C(\theta)$. 
Asymptotics of $\hat{\theta}_N$:

- If $H > \frac{3}{4}$, the GMM estimator does not converge to a multivariate normal law or even a second-chaos law.
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Numerical Details

- Approximation of $\rho_\theta(t) = 2\sigma^2 c_H \int_0^\infty \cos(tx) \frac{x^{1-2H}}{\lambda^2 + x^2} dx$:
  - Filon’s Method (integration of oscillatory integrands)
  - 4-5 precision digits.

- Simulation of fBm: Davies and Harte (1987).

- Approximation of fBm-OU process: the stationary fBm-OU satisfies:

$$X_{t_{i+1}} = e^{-\lambda \Delta} X_{t_i} + \sigma \int_{t_i}^{t_{i+1}} e^{-\lambda (t_i - u)} dB_u^H$$

where $X_0 \sim N(0, \rho_\theta(0))$.

- For comparison purposes we use the yuima R package of Brouste and Iacus (2012).

- Finite-difference filters.
Asymptotic behavior of $\sqrt{N}(\hat{\theta}_N - \theta)$

Figure: Asymptotic variance of $\sqrt{N}(\hat{H}_N - 0.62)$
Asymptotic behavior of $\sqrt{N}(\hat{\theta}_N - \theta)$

Figure: Asymptotic variance of $\sqrt{N}(\hat{\lambda}_N - 0.8)$
Numerical Details

- $(H, \lambda) = (0.37, 1.45)$
- $N = 1000, \alpha = 1$.
- $L = 3$ (filters’ maximum order)
- Number of repetitions in the sampling dist.: 100.
- Efficient matrix estimation: Continuous-updating estimator.
Sampling distribution of $\hat{H}_N$
Sampling distribution of $\hat{\lambda}_N$
Some statistics...

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<tbody>
<tr>
<td>Empirical MSE of $\hat{\theta}_N$</td>
<td>0.000155</td>
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<tr>
<td>Asymp. variance of $\hat{H}_N$ (empirical)</td>
<td>$3.11 \times 10^{-5}$</td>
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<td>Asymp. variance of $\hat{H}_N$ (theoretical)</td>
<td>$4.96 \times 10^{-5}$</td>
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<td>Asymp. variance of $\hat{\lambda}_N$ (empirical)</td>
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<td>Asymp. variance of $\hat{\lambda}_N$ (theoretical)</td>
<td>0.000193</td>
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Sampling distribution of $\hat{H}_N$

$(H, \lambda) = (0.8, 2)$ with 4 filters.
Sampling distribution of $\hat{\lambda}_N$

$(H, \lambda) = (0.8, 2)$ with 4 filters.
Future Work

- More accurate simulation of the fOU process?
- Asymptotic law of $\hat{\theta}_N$ when $H > \frac{3}{4}$? (fOU case)
Thanks!