Meyers inequality and Strong stability for stable-like operators

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Outline

- Part I: Preliminaries for Lévy processes with jumps;

- Part II: Meyers inequality and strong stability results for stable-like operators;
  - Caccioppoli inequality and Meyers inequality;
  - Strong stability of semigroups and heat kernels;
Part I: Preliminaries for Lévy processes with jumps
Continuous stochastic processes

Continuous stochastic processes have been widely applied in modeling in many areas; for example, the Ornstein-Uhlenbeck process and geometric Brownian motion.
Brownian motion

The most basic continuous stochastic process is the well-known Brownian motion.

Given a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), \(\{B_t\}_{t \geq 0}\) is a standard Brownian motion if:

1. \(B_0 = 0\), \(\mathbb{P}\)-a.s.;
2. \(B_t\) has continuous paths;
3. \(B_t - B_s\) has the normal distribution \(N(0, t - s)\) whenever \(s < t\);
4. \(B_t - B_s\) is independent of \(\mathcal{F}_s\) whenever \(s < t\);
Stochastic processes with jumps

However, continuous models suffer from some serious defects. For example, stock prices will at times decrease too fast to be followed by a geometric Brownian motion. A model that better fits the data is a geometric Brownian motion with jumps at random times.
The simplest jump stochastic process is the Poisson process.

Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, \{\(N_t\)\}_{t \geq 0} is a Poisson process with parameter \(\lambda > 0\) if

1. \(N_0 = 0\), \(\mathbb{P}\)-a.s.;
2. The paths of \(N_t\) are right continuous with left limits;
3. \(N_t - N_s\) is a Poisson r.v. with parameter \(\lambda(t - s)\) whenever \(s < t\);
4. \(N_t - N_s\) is independent of \(\mathcal{F}_s\) whenever \(s < t\);

Remark: A Poisson process can only have jumps of size 1.
**Lévy process**

Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), a process \(X\) is a Lévy process if

1. \(X(0) = 0\) \(\mathbb{P}\)-a.s.;
2. Stationary increments: \(X_t - X_s \overset{d}{=} X_{t-s}\) whenever \(s < t\);
3. Independent increments: \(X_t - X_s\) is independent of \(\sigma(X_r : r \leq s)\) whenever \(s < t\);
Brownian motions and Poisson processes are both Lévy process.

In this talk, we will study a class of pure jump Lévy processes—stable and stable-like processes.
Every Lévy process has a modification which is right continuous with left limits (càdlàg process). Let $X_{t-} = \lim_{s \uparrow t} X_s$ and $\Delta X_t = X_t - X_{t-}$.

Define

$$N(t, A) = \sum_{0 \leq s \leq t} 1_A(\Delta X_s):$$

the number of the jumps whose size is in the set $A$;

$$\nu(A) = \mathbb{E}(N(1, A)) :$$

the jump intensity measure of $X$ which is called the Lévy measure.
Three ways to study stochastic processes:

- Infinitesimal generators;
- Dirichlet forms;
- Stochastic differential equations (SDEs);
**Infinitesimal generator**

- Standard Brownian motion: $\mathcal{L}f(x) = \frac{1}{2} \Delta f(x)$;

We can generalize Brownian motion to other continuous diffusions. Two types of operators are common:
Non-divergence operators:

\[
\mathcal{L} f(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial f}{\partial x_i}(x);
\]

Divergence operators:

\[
\mathcal{L} f(x) = \sum_{i,j=1}^{d} \partial_i (a_{ij}(\cdot) \frac{\partial f}{\partial x_j}(\cdot))(x).
\]
• Poisson process with intensity $\lambda$: $\mathcal{L}f(x) = \lambda[f(x + 1) - f(x)];$

• Stable process:

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x + h) - f(x) - 1_{\{|h|<1\}} \nabla f \cdot h \right] \frac{c}{|h|^{d+\alpha}} \, dh,$$

where $c$ is a constant and $\alpha \in (0, 2).$
We can also generalize stable processes to stable-like processes.

- **Stable-like processes:**

  \[
  \mathcal{L}f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x + h) - f(x) - 1_{\{|h|<1\}} \nabla f \cdot h \right] n(x, dh),
  \]

  with some suitable conditions on \( n \).

For example: \( n(x, dh) = \frac{A(x, h)}{|h|^{d+\alpha}} \, dh \).
Dirichlet forms

To make sense of divergence operators when $a_{ij}$ are not differentiable, one looks at the following Dirichlet form:

$$E(f, g) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_j}(x) \, dx.$$ 

For jump processes, the Dirichlet forms one looks at are of the form

$$E(f, g) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))(g(y) - g(x)) \, J(dx, dy).$$

For example: $J(dx, dy) = \frac{A(x,y)}{|x-y|^{d+\alpha}} \, dx \, dy$ for $\alpha \in (0, 2)$. 
We can construct other stochastic processes via stochastic differential equations.

A one-dimensional SDE driven by Brownian motion is given by

\[ dX_t = \sigma(X_t) \, dB_t. \]

The solution to this equation is a diffusion process.
We can get jump type SDEs by replacing the Brownian motion by a Lévy process with jumps.

A one-dimensional SDE driven by a stable process of order $\alpha \in (0, 2)$ is given by

$$dX_t = \sigma(X_{t-}) \, dZ_t.$$ 

A càdlàg process $X$ is a solution if it satisfies the above equation.
Part II: Meyers inequality and stability results for stable-like operators
In this part, we study the stable-like processes associated with the operator

\[ \mathcal{L} f(x) = \int_{\mathbb{R}^d} \left[ f(y) - f(x) \right] \frac{A(x, y)}{|x - y|^{d+\alpha}} \, dy, \]  

(1)

where \( \alpha \in (0, 2) \), \( d \geq 2 \) and \( A(x, y) \) satisfies some suitable conditions.

The associated Dirichlet form is given by

\[ \mathcal{E}(u, v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))(v(y) - v(x)) \frac{A(x, y)}{|x - y|^{d+\alpha}} \, dy \, dx, \]

with domain \( \mathcal{D} = W^{\alpha, 2}(\mathbb{R}^d) \), a certain Sobolev-Besov space.
Assumptions on $A(x, y)$

We suppose:

1. Symmetry: $A(x, y) = A(y, x)$;
2. Boundedness: there exists a positive number $\Lambda$ such that $\Lambda^{-1} \leq A(x, y) \leq \Lambda$ for all $x, y \in \mathbb{R}^d$. 
Main Result: Meyers inequality

**Theorem**

Let \( u \) be the weak solution to \( \mathcal{L}u = h \) for \( h \in L^2 \). Then there exists \( p > 2 \) and a constant \( c_1 \) depending on \( \Lambda, p, d, \) and \( \alpha \) such that

\[
\| \Gamma u \|_{L^p(\mathbb{R}^d)} \leq c_1 \left( \mathcal{E}(u, u)^{\frac{1}{2}} + \| h \|_{L^2(\mathbb{R}^d)} + \| u \|_{L^2(\mathbb{R}^d)} \right),
\]

where \( \Gamma u(x) = \left( \int_{\mathbb{R}^d} \frac{(u(y) - u(x))^2}{|x-y|^{d+\alpha}} \, dy \right)^{\frac{1}{2}} \).

**Remark:** If \( u \in \mathcal{D}(\mathcal{L}) \), then there exists \( c_2 \), such that

\[
\| \Gamma u \|_{L^p(\mathbb{R}^d)} \leq c_2 \left( \| h \|_{L^2(\mathbb{R}^d)} + \| u \|_{L^2(\mathbb{R}^d)} \right).
\]
This is the analogue of the Meyers inequality for divergence form operators. An inequality of Meyers says that if $a_{ij}$ are uniformly elliptic and $u$ is a weak solution to $\mathcal{L}_d u = h$ for $h \in L^2$, then not only is $\nabla u$ locally in $L^2$ but it is locally in $L^p$ for some $p > 2$.

Remark: In our jump case, the “gradient” of $u$ is $\Gamma u$. 
In the continuous case, to derive the Meyers inequality one uses three main tools:

1. Caccioppoli inequality;
2. Sobolev-Poincaré inequality;
3. Reverse Hölder inequality;
Difficulties in the jump case

Our proof of the Meyers inequality also begins by proving a Caccioppoli inequality. However, as one might expect, our Caccioppoli inequality is not a local one, which requires the introduction of some new ideas, such as localization, use of the Hardy-Littlewood maximal function, and use of the Sobolev-Besov embedding theorem.
Caccioppoli inequality

Let $u \in W^{1,2}(\mathbb{R}^d)$ be the weak solution to

$$\mathcal{L}_d u(x) = h(x), \quad x \in \mathbb{R}^d,$$

where $\mathcal{L}_d$ is the divergence operator and $h \in L^2(\mathbb{R}^d)$.

Theorem (Caccioppoli inequality for divergence operators)

For all $x_0 \in \mathbb{R}^d$, and all $r, R$ with $0 < r < R < \infty$, we have

$$\int_{B_r(x_0)} |\nabla u|^2 \, dx \leq \frac{c}{(R - r)^2} \left[ \int_{B_R(x_0)} |u - u_R|^2 \, dx + \int_{B_R(x_0)} h^2(x) \, dx \right],$$

where $u_R = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(x) \, dx$. 
For our stable-like operators, we prove the Caccioppoli inequality for the weak solution to the equation

$$\mathcal{L} u(x) = h(x), \quad x \in \mathbb{R}^d \quad h \in L^2(\mathbb{R}^d).$$

(2)

A function $u \in \mathcal{W}^{\frac{\alpha}{2},2}(\mathbb{R}^d)$ is called a weak solution to (2) if

$$\mathcal{E}(u, v) = -(h, v) \quad \text{for all} \quad v \in \mathcal{W}^{\frac{\alpha}{2},2}(\mathbb{R}^d),$$

(3)

where $(h, v) = \int_{\mathbb{R}^d} h(x)v(x)\,dx.$
Our first key result shows the Caccioppoli inequality for the weak solution to equation (2):

**Theorem**

Let \( x_0 \in \mathbb{R}^d \). Suppose \( u(x) \) satisfies (3). There exists a constant \( c_1 \) depending only on \( \Lambda, \alpha, \) and \( d \) such that

\[
\int_{B_{R/2}} \int_{\mathbb{R}^d} (u(y) - u(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} \, dy \, dx
\]

\[
\leq c_1 \int_{\mathbb{R}^d} (u(y) - u_R)^2 \psi(y) \, dy + \int_{B_R} |h(y)(u(y) - u_R)| \, dy,
\]

where

\[
\psi(x) = R^{-\alpha} \vee \frac{R^d}{|x - x_0|^{d+\alpha}}.
\]
Lemma (Fractional Sobolev-Poincaré inequality)

If \( u \in W^{\frac{\alpha}{2}, q}(B_R) \), \( 1 < q < d \), then \( u \in L^p(B_R) \) for \( p = \frac{2dq}{2d - q\alpha} \), and there exists a constant \( c = c(d, q) \) such that

\[
\| u - u_R \|_{L^p(B_R)} \leq c \left[ \int_{B_R} \int_{B_R} \frac{(u(y) - u(x))^q}{|x - y|^{d + \frac{\alpha}{2}q}} \ dy \ dx \right]^{\frac{1}{q}}.
\]

Recall that

\[
\| f \|_{W^{\frac{\alpha}{2}, q}(\Omega)} = \| f \|_{L^q(\Omega)} + \left[ \int_\Omega \int_\Omega \frac{(f(y) - f(x))^q}{|x - y|^{d + \frac{\alpha}{2}q}} \ dy \ dx \right]^{\frac{1}{q}}.
\]
Based on the fractional Sobolev-Poincaré inequality and Hölder’s inequality, we obtain the following lemma:

**Lemma**

There exists $q_1 \in (1, 2)$ and a constant $c_1$ depending on $d, \alpha,$ and $q_1$ such that if $x_0 \in \mathbb{R}^d$ and $R > 0$, then

$$
\| u - u_R \|_{L^2(B_R)} \leq c_1 R^{(\alpha - \alpha_1)/2} \| \Gamma u \|_{L^{q_1}(B_R)},
$$

where $\alpha_1 = (2 - q_1) d / q_1$. 

Hardy-Littlewood maximal function

Given a locally integrable function $f(x)$ on $\mathbb{R}^d$, for each $r > 0$, we define

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy.$$ 

Remark: $M$ is a bounded operator from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$, i.e., there exists a finite constant $c = c(p, d) > 0$ such that

$$\|Mf\|_{L^p(\mathbb{R}^d)} \leq c\|f\|_{L^p(\mathbb{R}^d)} \text{ for } f \in L^p(\mathbb{R}^d) \text{ and } p > 1.$$
**Reverse Hölder inequality**

**Theorem (Reverse Hölder inequality)**

Let \( r > q > 1 \), \( 0 \leq g \in L^q(\Omega) \), \( 0 \leq f \in L^r(\Omega) \), and suppose for all \( B_R \subseteq \Omega \),

\[
\frac{1}{|B_r|} \int_{B_r(x_0)} g^q \, dx \leq c \left[ \frac{1}{|B_R|} \int_{B_R(x_0)} g \, dx \right]^q + \frac{1}{|B_R|} \int_{B_R(x_0)} f^q \, dx.
\]

Then \( g(x) \in L^{q+\varepsilon}(\Omega) \) for some \( \varepsilon > 0 \).

For example: if \( g(x) \in L^2(\Omega) \) and \( \|g(x)\|_{L^2(B_r)} \leq c \|g(x)\|_{L^1(B_R)} \), then \( g(x) \in L^{2+\varepsilon}(\Omega) \).
Put everything together

Based on Caccioppoli inequality, we have

$$\|\Gamma u\|_{L^2(B_{R/2})}^2 \leq c \int_{\mathbb{R}^d} (u(x) - u_R)^2 \psi(x) \, dx + \int_{B_R} |h(x)(u(x) - u_R)| \, dx$$

$$\leq c \int_{B_R} (u(x) - u_R)^2 \, dx + c \int_{B_{Rc}} u(x)^2 \psi(x) \, dx$$

$$+ c \int_{B_{Rc}} u_R^2 \psi(x) \, dx + \int_{B_R} |h(x)(u(x) - u_R)| \, dx$$

$$=: J_1 + J_2 + J_3 + J_4.$$
For term $J_1$, we apply the previous Lemma:

$$J_1 = \int_{B_R} (u(x) - u_R)^2 \, dx \leq c \left( \int_{B_R} \Gamma u(x)^{q_1} \, dx \right)^{\frac{2}{q_1}}$$
For term $J_2$, $J_3$ and $J_4$, we take use of Hardy-Littlewood maximal function as the main tool:

$$J_2 = \int_{B_R^c} u(x)^2 \psi(x) \, dx \leq c \, M(u^2)(y);$$

$$J_3 = \int_{B_R^c} u_R^2 \psi(x) \, dx \leq c \, M(u^2)(y);$$

$$J_4 = \int_{B_R} |h(x)(u(x) - u_R)| \, dx \leq c \int_{B_R} |h(x)| \, Mu(x) \, dx.$$
Combining all these bounds and integrating over $y \in B_R$ gives

$$
\int_{B_{R/2}} \Gamma u(x)^2 \, dx \leq c \left( \int_{B_R} \Gamma u(x)^{q_1} \, dx \right)^{\frac{2}{q_1}}
$$

$$
+ c \int_{B_R} M(u^2)(x) \, dx + c \int_{B_R} |h(x)|Mu(x) \, dx.
$$
Let

\[ g(x) = \Gamma u(x)^{q_1} \]

and

\[ f(x) = \left( M(u^2)(x) + |h(x)|Mu(x) \right)^{\frac{q_1}{2}}, \]

then

\[
\frac{1}{|B(x_0, R)|} \int_{B(x_0, R/2)} g^{\frac{2}{q_1}}(x) \, dx \\
\leq c \left( \frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} g(x) \, dx \right)^{\frac{2}{q_1}} + c \frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} f^{\frac{2}{q_1}}(x) \, dx.
\]

Here is where Reverse Hölder comes into play.
One of the applications of Meyers inequality is to obtain the stability of solutions to $\mathcal{L}_d u = h$. Suppose one perturbs the coefficients $a_{ij}$ slightly. How does this affect the semigroup and fundamental solution associated with the operator $\mathcal{L}_d$?
In the diffusion case, this question has been answered by Chen, Qian, Hu and Zheng (1998).
Based on our main result — the Meyers inequality for stable-like operators \( \mathcal{L} \), strong stability results can be proved along the lines of proof in the diffusion case.
Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be the stable-like operators of the form $(1)$. Let $p(t, x, y)$ and $\tilde{p}(t, x, y)$ be the corresponding heat kernels, $P_t$ and $\widetilde{P}_t$ be the corresponding semigroups.
Suppose \( d > \alpha \). There exist \( q \geq 2d/\alpha \) and a constant \( c_1 \) depending on \( \Lambda, d, \alpha, \) and \( q \) such that if \( f \in L^2(\mathbb{R}^d) \), then

\[
\| P_t f - \tilde{P}_t f \|_{L^2}^2 \leq c_1 t^{-\frac{d}{2q\alpha}} \| G \|_{L^2} \| f \|_{L^2}^2,
\]

where \( G(x) = \sup_{y \in \mathbb{R}^d} |\tilde{A}(x, y) - A(x, y)| \).
Sketch of the proof

Let $u = P_t f - \tilde{P}_t f$. We can write

$$\|P_t f - \tilde{P}_t f\|_{L^2}^2 = (P_t f - \tilde{P}_t f, u)$$

$$= \int_0^t \frac{d}{ds} (P_s \tilde{P}_{t-s} f, u) \, ds.$$ 

$$= \int_0^t (-\mathcal{E}(\tilde{P}_{t-s} f, P_s u) + \tilde{\mathcal{E}}(\tilde{P}_{t-s} f, P_s u)) \, ds$$

$$\leq c \int_0^t \left[ \tilde{\mathcal{E}}(\tilde{P}_{t-s} f, \tilde{P}_{t-s} f) \right]^{1/2} \|\Gamma(P_s u)(x)\|_{2p} \|G(x)\|_{2q} \, ds$$

$$=: c \cdot \int_0^t I \times II \, ds \cdot \|G(x)\|_{2q}.$$
For I: We can bound this part by using spectral representation theory;

For II: This is where we use our Meyers inequality.
We make use of the following two results for heat kernels:

**Theorem (Chen and Kumagai, 2003)**

There are constants $c_1, c_2 > 0$ that depend on $d, \alpha$ such that

$$c_1 \min \left\{ t^{-\frac{d}{\alpha}}, \frac{t}{|x - y|^{d+\alpha}} \right\} \leq p(t, x, y) \leq c_2 \min \left\{ t^{-\frac{d}{\alpha}}, \frac{t}{|x - y|^{d+\alpha}} \right\}, \forall x, y \in \mathbb{R}^d.$$

**Theorem (Chen and Kumagai, 2003)**

There exist $c_3 > 0$, and $\gamma > 0$ such that

$$|p(t, x_1, y_1) - p(t, x_2, y_2)| \leq c_3 t^{-\frac{d+\gamma}{\alpha}} (|x_1 - x_2| + |y_1 - y_2|)^\gamma.$$
Strong stability of $p(t, x, y)$

**Theorem**

Let $t > 0$. There exist $q > 1$ and a constant $c_1$ depending on $t, \Lambda, \gamma, d, \alpha$, and $q$ such that for any $x, y \in \mathbb{R}^d$

$$|p(t, x, y) - \tilde{p}(t, x, y)| \leq c_1 \|G\|_{2q}^{\frac{\gamma}{2(d+\gamma)}}.$$
Theorem

Let $t > 0$. There exist $q > 1$ and a constant $c_2$ depending on $t, \Lambda, \gamma, d, \alpha,$ and $q$ such that for any $p \in [1, \infty]$, we have

$$\|P_tf - \tilde{P}_tf\|_{L^p} \leq c_2 \|G\|_{2q}^{\frac{\gamma\alpha}{2(d+\gamma)(d+\alpha)}} \|f\|_{L^p}.$$
Thank you!