The Probability That A 2×2 Matrix All Real Eigenvalues

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1 Introduction

Matrix are important in the fields of mathematics and statistics, especially linear algebra, so Andrew Hetzel, Jay Liew and Kent Morrison tried to find the probability that a 2×2 matrix with integer entries in the interval[-k,k] is diagonalizable[1]. They stated that the probability of diagonalizability is $\frac{49}{72}$. When they proved this, they considered the discriminant of characteristic polynomial of a random matrix. However, this method is not useful for higher dimemsional matrix such as 3×3 matrix, so in this paper we tried to find a more general method to compute this probability, a method related to the joint density of trace and determinant of a 2×2 matrix.

2 Joint Density of Determinant and Trace

For a 2×2 matrix $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, let $A = X + W$ be the trace and $B = XW - YZ$ be determinant. For getting the joint density of $(A, B) = (X + W, XW - YZ)$, we consider to use the method of joint probability distribution of functions of random variables [2]. There are four random variables X, Y, Z and W, for simplisity, we tried to find a method to reduce this problem in four dimensions to two dimensions at first. Therefore we fix YZ, then we can find the conditional probability of $A = X + W$ and $B = XW - YZ$ with fixed YZ. Then if we integrate out YZ later, we can get the joint density of $(A, B) = (X + W, XW - YZ)$.

Theorem 2.1. *For a 2×2 matrix $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, assume X, Y, Z, W are from continuous uniform distribution in the interval [-1,1], then the joint density of the trace (A) and determinant (B) is*

$$f_{A,B}(a,b) = \begin{cases} g_1(a,b) & \text{if} \quad |a| - 2 < b < \frac{a^2-4}{4} \\ g_2(a,b) & \text{if} \quad \frac{a^2-4}{4} < b < |a| - 1 \\ g_3(a,b) & \text{if} \quad |a| - 1 < b < \frac{a^2}{4} \\ g_4(a,b) & \text{if} \quad \frac{a^2}{4} < b < |a| \\ g_5(a,b) & \text{if} \quad |a| < b < \frac{a^2+4}{4} \\ 0 & \textit{otherwise} \end{cases}$$

where

$$g_1(a,b) = \frac{1}{8}\left(-2\sqrt{-4 + a^2 - 4b} + 2(2 - |a|) - (2 - |a|)\log[-1 - b + |a|]\right)$$

$$g_2(a,b) = \frac{1}{8}\left(2(2-|a|) - (2-|a|)\log[-1-b+|a|] - \sqrt{a^2-4b}\log\left[-\frac{a^2+(2-|a|)\sqrt{a^2-4b}-2(-1+b+|a|)}{2(1+b-|a|)}\right]\right)$$

$$g_3(a,b) = \frac{1}{8}\left(-\sqrt{a^2-4b}\log\left[\frac{2+a^2-2b+\sqrt{a^2-4b}(2-|a|)-2|a|}{2+2b-2|a|}\right] - (2-|a|)(-2+\log[1+b-|a|])\right)$$

$$g_4(a,b) = \frac{1}{8}\left(-2\sqrt{4b-a^2}\arctan\left[\frac{2-|a|}{\sqrt{4b-a^2}}\right] + 2(2-|a|) - (2-|a|)\log[1+b-|a|]\right)$$

$$g_5(a,b) = \frac{1}{4}\left(\sqrt{4+a^2-4b} - \sqrt{4b-a^2}\arctan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right]\right)$$

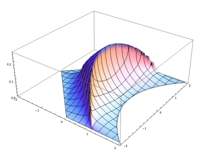

Figure 1: joint density of trace and determinant

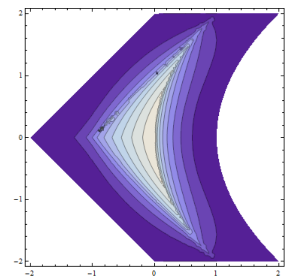

Figure 2: Contour plot of joint density of trace and determinant

Proof. Fix Y and Z, let $S = YZ$. Then we have a system of equations of trace (labeled by a) and determinant (labeled by b)

$$\begin{cases} a = x + w \\ b = xw - s \end{cases}$$

Solve for x and w in terms of a, b and s (because s is a constant here), there are two solutions:

- solution 1: $\begin{cases} x = \frac{a-\sqrt{a^2-4b-4s}}{2} \\ w = \frac{a+\sqrt{a^2-4b-4s}}{2} \end{cases}$

- solution 2: $\begin{cases} x = \frac{a+\sqrt{a^2-4b-4s}}{2} \\ w = \frac{a-\sqrt{a^2-4b-4s}}{2} \end{cases}$

Recall that x and w are in $[-1,1]$, so $\frac{a\pm\sqrt{a^2-4b-4s}}{2}$ is in $[-1,1]$. Also it is easy to check that $a^2-4b-4s \geq 0$, so $\frac{a\pm\sqrt{a^2-4b-4s}}{2}$ is real. Because x in solution 1 is the same as w in solution 2, and so is w in solution 1, $|J(x,w)|^{-1}$ and $f_{X,W}(x,w)$ are the same no matter which solutions we take, thus the joint density of $(A, B|S)$ can be obtained by using the joint density of (X, W) such that

$$f_{A,B,S}(a,b|s) = 2f_{X,W}(x,w)|J(x,w)|^{-1} \tag{1}$$

Without loss of generality, we use sulution 1: $x = \frac{a-\sqrt{a^2-4b-4s}}{2}$ and $w = \frac{a+\sqrt{a^2-4b-4s}}{2}$. Then

$$J(x,w) = \begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial w} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ w & x \end{vmatrix} = x - w \tag{2}$$

Since X and W have continuous uniform distribution independently in the interval [-1,1], the joint density of (X, W) is

$$f_{X,W}(x,w) = \begin{cases} f_X(x) f_W(w) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} & \text{if } x, w \in [-1,1] \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Plugging (2) and (3) into (1) and substituting solution 1 :

$$\begin{aligned} f_{A,B,S}(a,b|s) &= 2 f_{X,W}(x,w) |J(x,w)|^{-1} \\ &= \frac{1}{2|x-w|} \\ &= \frac{1}{2\sqrt{a^2 - 4b - 4s}} \end{aligned}$$

Therefore the joint density of the trace (A) and determinant (B) is given by

$$f_{A,B}(a,b) = \int_s f_{A,B,S}(a,b|s) f_S(s)\, ds \tag{4}$$

Note: Here are two lemmas where the proofs of them are in section 4.
Lemma 4.1:
For two independent random variables Y and Z from continous uniform distribution between -1 and 1, the distribution of YZ is

$$f_{YZ}(s) = -\frac{\log[|s|]}{2}$$

where $s \in [-1, 1]$.
Lemma 4.2:
For a 2×2 matrix $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$, assume x, y, z, w are in the interval [-1,1] and $s = yz$,then the region for s, trace (a) and determinant (b) is given by
(i) $|a| - 1 - b \le s \le \frac{a^2-4b}{4}$
(ii) $-1 < s < 1$

From above lemmas, the joint distribution of (A, B) which are trace and determinant of a 2×2 matrix becomes

$$f_{A,B}(a,b) = \int_s f_{A,B,S}(a,b|s) f_S(s)\, ds = \int_{|a|-1-b \le s \le \frac{a^2-4b}{4}, -1 \le s \le 1} -\frac{\log[|s|]}{4\sqrt{a^2-4b-4s}}\, ds \tag{5}$$

For getting $f_{A,B}(a,b)$ in equation (5), we have five cases:

I. $0 < |a| - 1 - b < 1 < \frac{a^2-4b}{4}$

II. $0 < |a| - 1 - b < \frac{a^2-4b}{4} < 1$

III. $-1 < |a| - 1 - b < 0 < \frac{a^2-4b}{4} < 1$

IV. $-1 < |a| - 1 - b < \frac{a^2-4b}{4} < 0$

V. $|a| - 1 - b < -1 < \frac{a^2-4b}{4} < 0$

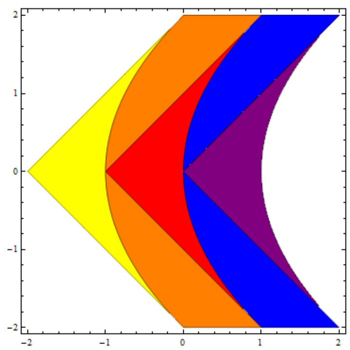

Figure 3: Contour plot of joint density of trace and determinant, where yellow is the case I, orange is the case II, red is case III, blue is case IV and purple is case V

Note: Now we introduce a lemma to help us get $f_{A,B}(a,b)$. Again, the proof of this lemma is in section 4.

Lemma 2.2. *Fix a and b. Given that*

$$f(a,b,s) = \begin{cases} f_1(a,b,s) & \text{if} \quad 0 < 4s < a^2 - 4b \\ f_2(a,b,s) & \text{if} \quad 4s < 0 < a^2 - 4b \\ f_3(a,b,s) & \text{if} \quad 4s < a^2 - 4b < 0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$f_1(a,b,s) = \log[2]\sqrt{a^2-4b-4s} - \frac{\sqrt{a^2-4b-4s}\log[4s]}{2} + \sqrt{a^2-4b-4s} - \frac{\sqrt{a^2-4b}}{2}\log\left[\frac{\sqrt{a^2-4b-4s}+\sqrt{a^2-4b}}{-\sqrt{a^2-4b-4s}+\sqrt{a^2-4b}}\right]$$

$$f_2(a,b,s) = \log[2]\sqrt{a^2-4b-4s} - \frac{\sqrt{a^2-4b-4s}\log[-4s]}{2} + \sqrt{a^2-4b-4s} - \frac{\sqrt{a^2-4b}}{2}\log\left[\frac{\sqrt{a^2-4b-4s}+\sqrt{a^2-4b}}{\sqrt{a^2-4b-4s}-\sqrt{a^2-4b}}\right]$$

$$f_3(a,b,s) = \log[2]\sqrt{a^2-4b-4s} - \frac{\sqrt{a^2-4b-4s}\log[-4s]}{2} + \sqrt{a^2-4b-4s} - \sqrt{4b-a^2}\arctan\left[\sqrt{\frac{a^2-4b-4s}{4b-a^2}}\right]$$

Then

$$\frac{\partial}{\partial s} f(a,b,s) = \frac{\log[|s|]}{\sqrt{a^2-4b-4s}}$$

Remark 2.3. It is easy to check this lemma by finding derivatives directly for interiors of each case. But it is important to note that it is defined for boundary points so it is continuous (we will left them to you to check out). And we will use this lemma to compute the integral $\int_s \frac{\log[|s|]}{\sqrt{a^2-4b-4s}}\, ds$.

With this lemma, let us get the $f_{A,B}(a,b)$ case by case.

In case (I):

$0 < |a| - 1 - b < 1 < \frac{a^2-4b}{4}, \quad |a| - 1 - b < s < \frac{a^2-4b}{4}$ and $-1 < s < 1$, so $0 < |a| - 1 - b < s < 1$ and domian of a and b is $|a| - 2 < b < \frac{a^2-4}{4}, \quad -2 < a < 2$. By using the formula in Lemma 2.2

when s is positive.

$$
\begin{aligned}
f_{A,B}(a,b) &= \int_{|a|-1-b}^{1} -\frac{\log[s]}{4\sqrt{a^2-4b-4s}}\,ds \\
&= -\frac{1}{4}\left[f_1(a,b,1) - f_1(a,b,|a|-1-b)\right] \\
&= \frac{1}{8}\left(-2\sqrt{-4+a^2-4b} + 2(2-|a|) - (2-|a|)\log[-1-b+|a|]\right) \\
&= g_1(a,b) \qquad (6)
\end{aligned}
$$

In case (II):
$0 < |a|-1-b < \frac{a^2-4b}{4} < 1$, $|a|-1-b < s < \frac{a^2-4b}{4}$ and $-1 < s < 1$, so $|a|-1-b < s < \frac{a^2-4b}{4}$ and domian of a and b is $\frac{a^2-4}{4} < b < |a|-1$, $-2 < a < 2$. By using the formula in Lemma 2.2 when s is positive.

$$
\begin{aligned}
f_{A,B}(a,b) &= \int_{|a|-1-b}^{\frac{a^2-4b}{4}} -\frac{\log[s]}{4\sqrt{a^2-4b-4s}}\,ds \\
&= -\frac{1}{4}\left[f_1(a,b,\frac{a^2-4b}{4}) - f_1(a,b,|a|-1-b)\right] \\
&= \frac{1}{8}(2(2-|a|) - (2-|a|)\log[-1-b+|a| \\
&\quad - \sqrt{a^2-4b}\log\left[-\frac{a^2+(2-|a|)\sqrt{a^2-4b}-2(-1+b+|a|)}{2(1+b-|a|)}\right]) \\
&= g_2(a,b) \qquad (7)
\end{aligned}
$$

In case (III):
$-1 < |a|-1-b < 0 < \frac{a^2-4b}{4} < 1$ $|a|-1-b < s < \frac{a^2-4b}{4}$ and $-1 < s < 1$, so $|a|-1-b < s < \frac{a^2-4b}{4}$ and domain of a and b is $|a|-1 < b < \frac{a^2}{4}$, $-2 < a < 2$. By using the formula in Lemma 2.2 when s is positive and when s is negative, a^2-4b is positive.

$$
\begin{aligned}
f_{A,B}(a,b) &= \int_{|a|-1-b}^{\frac{a^2-4b}{4}} -\frac{\log[|s|]}{4\sqrt{a^2-4b-4s}}\,ds \\
&= \int_{|a|-1-b}^{0} -\frac{\log[-s]}{4\sqrt{a^2-4b-4s}}\,ds + \int_{0}^{\frac{a^2-4b}{4}} -\frac{\log[s]}{4\sqrt{a^2-4b-4s}}\,ds \\
&= -\frac{1}{4}\left[f_1(a,b,0) - f_1(|a|-1-b) + f_2(a,b,\frac{a^2-4b}{4}) - f_2(a,b,0)\right] \\
&= \frac{1}{8}\left(-\sqrt{a^2-4b}\log\left[\frac{2+a^2-2b+(2-|a|)\sqrt{a^2-4b}-2|a|}{2+2b-2|a|}\right] - (2-|a|)(-2+\log[1+b-|a|])\right) \\
&= g_3(a,b) \qquad (8)
\end{aligned}
$$

In case (IV):
$-1 < |a|-1-b < \frac{a^2-4b}{4} < 0$, $|a|-1-b < s < \frac{a^2-4b}{4}$ and $-1 < s < 1$, so $|a|-1-b < s < \frac{a^2-4b}{4}$ and domain of a and b is $\frac{a^2}{4} < b < |a|$ and $-2 < a < 2$. Because s and a^2-4b are negative, we

use the formula in Lemma 2.2 when s and a^2-4b are neagative to get $f_{A,B}(a,b)$.

$$\begin{aligned}
f_{A,B}(a,b) &= \int_{|a|-1-b}^{\frac{a^2-4b}{4}} -\frac{\log[-s]}{4\sqrt{a^2-4b-4s}}\,ds \\
&= -\frac{1}{4}\left[f_3(a,b,\frac{a^2-4b}{4}) - f_3(a,b,|a|-1-b)\right] \\
&= \frac{1}{8}\left(-2\sqrt{4b-a^2}ArcTan\left[\frac{2-|a|}{\sqrt{4b-a^2}}\right] + 2(2-|a|) - (2-|a|)\log[1+b-|a|]\right) \\
&= g_4(a,b) \qquad (9)
\end{aligned}$$

In case (V) :
we have $|a|-1-b < -1 < \frac{a^2-4b}{4} < 0$, $|a|-1-b < s < \frac{a^2-4b}{4}$ and $-1 < s < 1$, so $-1 < s < \frac{a^2-4b}{4}$ and domain of a and b is $|a| < b < \frac{a^2+4}{4}$ and $-2 < a < 2$. Because s and a^2-4b are negative, we use the formula in Lemma 2.2 when s and a^2-4b are negative.

$$\begin{aligned}
f_{A,B}(a,b) &= \int_{-1}^{\frac{a^2-4b}{4}} -\frac{\log[-s]}{4\sqrt{a^2-4b-4s}}\,ds \\
&= -\frac{1}{4}\left[f_3(a,b,\frac{a^2-4b}{4}) - f_3(a,b,-1)\right] \\
&= \frac{1}{4}\left(\sqrt{4+a^2-4b} - \sqrt{4b-a^2}ArcTan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right]\right) \\
&= g_5(a,b) \qquad (10)
\end{aligned}$$

□

3 The Probability That a 2×2 Matrix Have All Real Eigenvalues

After getting the joint density of trace and determinant of a random 2×2 matrix, we show how a random 2×2 matrix has all real eigenvalues.

Corollary 3.1. *For a 2×2 matrix $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, assume X,Y,Z,W are from continuous uniform distribution in the interval [-1,1], then the probability that this matrix is diagonalizable over $\mathbb{R}$ is $\frac{49}{72}$.*

Proof. For the a 2×2 matrix $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, the characteristic polynomial is $\lambda^2-(X+W)\lambda+(WX-YZ) = \lambda^2-(trace)\lambda+(determinant)$. Let A be the trace and B be the determinant, then the discriminant of $\lambda^2-(X+W)\lambda+(XW-YZ) = \lambda^2-(trace)\lambda+(determinant) = 0$ is $(X+W)^2-4(XW-YZ) = A^2-4B$. Therefore the discriminant (A^2-4B) should be positive, for all real eigenvalues, which is the same as $B < \frac{A^2}{4}$. Thus the probability that a 2×2 matrix has all real eigenvalues is given by

$$\begin{aligned} P(\text{all eigenvalues are real}) &= P(B < \frac{A^2}{4}) \\ &= 1 - P(B > \frac{A^2}{4}) \\ &= 1 - \iint_{b > \frac{a^2}{4}} f_{A,B}(a,b)\, db\, da \end{aligned} \tag{11}$$

From Theorem 2.1, we know that $b > \frac{a^2}{4}$ has two parts: $\frac{a^2}{4} < b < |a|$ and $|a| < b < \frac{a^2+4}{4}$.

$$\iint_{b > \frac{a^2}{4}} f_{A,B}(a,b)\, db\, da = \iint_{\frac{a^2}{4}}^{|a|} f_{A,B}(a,b)\, db\, da + \iint_{|a|}^{\frac{a^2+4}{4}} f_{A,B}(a,b)\, db\, da \tag{12}$$

For simplicity, we can integrate these two integrals one by one.
For the first integral,

$$\iint_{\frac{a^2}{4}}^{|a|} f_{A,B}(a,b)\, db\, da = 2\int_0^2 \int_{\frac{a^2}{4}}^{a} f_{A,B}(a,b)\, db\, da \tag{13}$$

From Theorem 2.1, integral in (13) becomes

$$\begin{aligned} &\frac{1}{4}\int_0^2 \int_{\frac{a^2}{4}}^{a} -2\sqrt{4b-a^2}\arctan\left[\frac{2-a}{\sqrt{4b-a^2}}\right] + 2(2-a) - (2-a)\log[1+b-a]\, db\, da \\ &= \frac{1}{4}\int_0^2 \int_{\frac{a^2}{4}}^{a} -2\sqrt{4b-a^2}\arctan\left[\frac{2-a}{\sqrt{4b-a^2}}\right] db\, da \\ &\quad + \frac{1}{4}\int_0^2 \int_{\frac{a^2}{4}}^{a} 2(2-a)\, db\, da \\ &\quad - \frac{1}{4}\int_0^2 \int_{\frac{a^2}{4}}^{a} (2-a)\log[1+b-a]\, db\, da \end{aligned} \tag{14}$$

There are two lemmas we need for the first and third integrals above (the second one can be obtained by general calculus method).
Lemma 4.3:

$$2\int_0^2 \int_{\frac{a^2}{4}}^{a} \sqrt{4b-a^2}\arctan\left[\frac{2-a}{\sqrt{4b-a^2}}\right] db\, da = \frac{1}{4}(-4+\pi^2)$$

Lemma 4.4:

$$\int_0^2 \int_{\frac{a^2}{4}}^{a} (2-a)\log[1+b-a]\, db\, da = -\frac{1}{2}$$

Then sum of three integrals in (14) is

$$\frac{1}{4}\left(-\frac{1}{4}(-4+\pi^2)\right) + \frac{1}{4}2 - \frac{1}{4}\left(-\frac{1}{2}\right) = \frac{1}{4}\left(\frac{5}{2} + \frac{1}{4}(4-\pi^2)\right) \tag{15}$$

For the second integral in (12),

$$\iint_{|a|}^{\frac{a^2+4}{4}} f_{A,B}(a,b)\,db\,da = 2\int_0^2 \int_a^{\frac{a^2+4}{4}} f_{A,B}(a,b)\,db\,da \tag{16}$$

From Theorem 2.1, integral in (16) becomes

$$\begin{aligned}
&\frac{1}{2}\int_0^2 \int_a^{\frac{a^2+4}{4}} \sqrt{4+a^2-4b} - \sqrt{4b-a^2}\arctan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right] db\,da \\
&= \frac{1}{2}\times\frac{2}{3} - \frac{1}{2}\int_0^2 \int_a^{\frac{a^2+4}{4}} \sqrt{4b-a^2}\arctan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right] db\,da
\end{aligned} \tag{17}$$

For getting the final result, we introduce lemma 4.5 such that

$$\int_0^2 \int_a^{\frac{a^2+4}{4}} \sqrt{4b-a^2}\arctan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right] db\,da = \frac{16}{9} - \frac{\pi^2}{8}$$

Note: you can simply find $\int_0^2 \int_a^{\frac{a^2+4}{4}} \sqrt{4+a^2-4b}\,db\,da$ by using the general method in calculus. Then integral (17) is given by

$$\frac{1}{3} - \frac{1}{2}\left(\frac{16}{9} - \frac{\pi^2}{8}\right) = -\frac{5}{9} + \frac{\pi^2}{16}$$

Combine with the result of first integral (16), the probability in (11) is

$$1 - \frac{1}{4}\left(\frac{5}{2} + \frac{1}{4}(4-\pi^2)\right) - \left(-\frac{5}{9} + \frac{\pi^2}{16}\right) = \frac{49}{72}$$

□

4 Important Lemmas With Their Proofs

You may notice that we used many lemmas in the previous proof, now we state those lemma and their proofs here.

Lemma 4.1. *For two independent random variables Y and Z from continous uniform distribution between -1 to 1, the distribution of yz is*

$$f_{YZ}(s) = -\frac{\log[|s|]}{2}$$

where $s \in [-1, 1]$.

Proof. Y (or Z) is from continuous uniform distribution between -1 and 1, so its probability density function is $f(y) = \frac{1}{2}$ and $-1 < yz < 1$. Because Y and Z are independent random variables, the joint density of (Y, Z) is the product of their probability density functions, which

is $f(y)f(z) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ on $[-1,1] \times [-1,1]$.
Let $s = yz$ and $F_{YZ}(s)$ be the cumulatice distribution function of YZ. Then

$$F_{YZ}(s) = P(yz < s) = \iint\limits_{\substack{yz<s \\ y\in[-1,1] \\ z\in[-1,1]}} f(y,z)\,dy\,dz = \iint\limits_{\substack{yz<s \\ y\in[-1,1] \\ z\in[-1,1]}} \frac{1}{4}\,dy\,dz$$

For simplicity, we evaluate the integral by two cases: (i) $s \in [0,1]$, (ii) $s \in [-1,0]$.
In case (i):

$$F_{YZ}(s) = \frac{1}{2}\left(1 + s - s\log[s]\right)$$

And by symmetry, in case (ii), we have $F_{YZ}(s) = \frac{1}{2}\left(1 + s - s\log[-s]\right)$.
To combine two cases, the cumulative distribution function of YZ is $F_{YZ}(s) = \frac{1}{2}\left(1 + s - s\log[|s|]\right)$.
Because the probability density function of YZ is the derivative of cumulative distribution function of YZ respect to s, the density of YZ is $f_{YZ}(s) = -\frac{\log[|s|]}{2}$ where $S \in [-1,1]$. $\square$

Lemma 4.2. *let* $A = trace\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = x + w$, $B = det\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = XW - YZ$ *and* $S = YZ$. *Assume* X, Y, Z, W *are in the interval [-1,1], then the joint density* $f_{A,B,S}(a,b,s)$ *is nonzero if and only if*

$$\begin{cases} |a| - b - 1 \le s \le \frac{a^2-4b}{4} \\ -1 \le s \le 1 \end{cases}$$

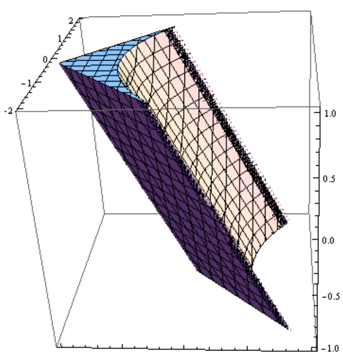

Figure 4: Region of (a,b,s)

Proof. First, any value of $s \in (-1,1)$ is possible since $s = yz$ where $y, z \in (-1,1)$.
For s fixed, the values of determinant and trace are

$$\begin{cases} a = x + w \\ b = xw - s \end{cases}$$

So the domain of joint density $f_{A,B,S}(a,b,s)$ can be obtained by fixing s then shifting from -1 to 1. When $s = 0$, $a = x + w$ and $b = xw$ where $x, w \in (-1,1)$. So the domain in terms of x and w is as followed:

For transforming the interior points on xw-plane, we look at the lines $w = x + c$ where $c \in (-1,1)$. Then interior points can be transformed by $(x, x+c) \to (x^2 + cx, 2x + c) = (b,a)$ where
(I) $x \in (-1, 1-c)$ and $c \in (0,1)$;

(II) $x \in (-1-c, 1)$ and $c \in (-1, 0)$.
In case (I), all points should be on the left side of $\frac{a^2-4b}{4}$. With the boundaries, the graph is shown below:
Similarly, in case (II) we have the same region as in case (I).

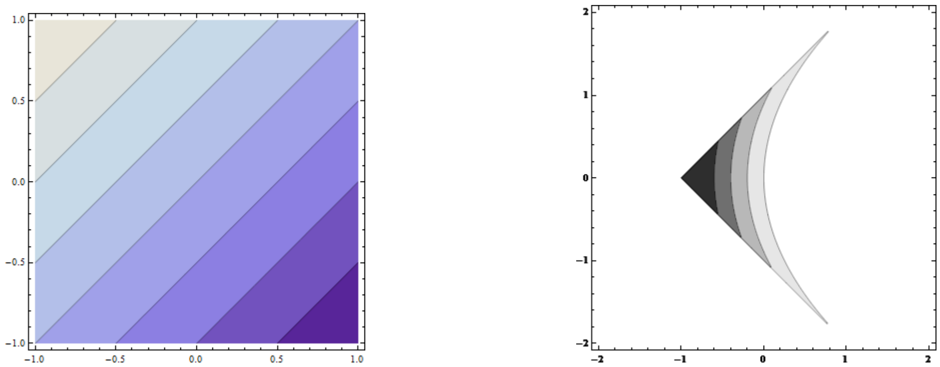

Figure 5: On the left are the lines $w = x + c$. On the right are the curves $(b, a) = (2x+c, x^2+cx)$.

Remark: In case (I), we transformed points below the diagonal. In case (II), we transformed points above the diagonal. But both cases have the same region after transformation, so the mapping is actually 2 to 1.
Now we transform the domain for nonzero $f_{A,B,S}(a, b, s)$ with fixed $s = 0$ on xw-plane into ba-plane such that $|a| - b - 1 \leq s \leq \frac{a^2-4b}{4}$.

□

Lemma 4.3.

$$2\int_0^2 \int_{\frac{a^2}{4}}^{a} \sqrt{4b-a^2}\arctan\left[\frac{2-a}{\sqrt{4b-a^2}}\right] db\, da = \frac{1}{4}(-4+\pi^2)$$

Proof. Let $u = \frac{1}{6}(4b-a^2)^{\frac{3}{2}}$, then $du = \sqrt{4b-a^2}\, db$. By substitution, the integral becomes

$$2\int_0^2 \int_0^{\frac{1}{6}(4a-a^2)^{\frac{3}{2}}} \arctan\left[(2-a)6^{-\frac{1}{3}}u^{-\frac{1}{3}}\right] du\, da \tag{18}$$

Now we introduce a derivative that you can easily check.
Derivative:
If $f(c, u) = \frac{1}{2}\left(-c^3\log\left[c^2+u^{\frac{2}{3}}\right] + cu^{\frac{2}{3}} + 2u\arctan\left[cu^{-\frac{1}{3}}\right]\right)$ on $u, c > 0$, then

$$\frac{\partial}{\partial u} f(c, u) = \arctan\left[cu^{-\frac{1}{3}}\right] \tag{19}$$

Because of this derivative, the integral (18) becomes

$$-\frac{4}{9}\log\left[\frac{4}{3}\right] + \frac{2}{3} + \frac{1}{3}\int_0^2 (4a-a^2)^{\frac{3}{2}}\arctan\left[(2-a)(4a-a^2)^{-\frac{1}{2}}\right] da + \frac{1}{3} - \frac{4}{9}\log\left[\frac{4}{3}\right] \tag{20}$$

To evaluate the integral in (20), let $u = \arctan\left[\frac{2-a}{\sqrt{4a-a^2}}\right]$ and $dv = -(4a-a^2)^{\frac{3}{2}}\, da$, then by integral by part, we have

$$\begin{aligned}
&\int -(4a-a^2)^{\frac{3}{2}}\arctan\left[(2-a)(4a-a^2)^{-\frac{1}{2}}\right]\,da\\
&=\arctan\left[(2-a)(4a-a^2)^{-\frac{1}{2}}\right]\left(\frac{(2-a)(4a-a^2)^{\frac{3}{2}}}{4}+\frac{3(2-a)(4a-a^2)^{\frac{1}{2}}}{2}+6\arcsin\left[\frac{2-a}{2}\right]\right)\\
&\quad-\frac{(2-a)^2}{2}+\frac{(2-a)^4}{16}-\frac{3(2-a)^2}{4}+6\int\arcsin\left[\frac{2-a}{2}\right](4a-a^2)^{-\frac{1}{2}}\,da
\end{aligned}\tag{21}$$

To find $-6\int\arcsin\left[\frac{2-a}{2}\right](4-(2-a)^2)^{-\frac{1}{2}}\,da$ in (21), substitute $g=\arcsin\left[\frac{2-a}{2}\right]$ and $dg=\frac{-1}{\sqrt{4a-a^2}}\,da$,

$$\begin{aligned}
-6\int\arcsin\left[\frac{2-a}{2}\right](4a-a^2)^{-\frac{1}{2}}\,da &= 6\int g\,dg\\
&=3\left(\arcsin\left[\frac{2-a}{2}\right]\right)^2
\end{aligned}$$

Therefore above equation (21) is given by

$$\begin{aligned}
&\int -(4a-a^2)^{\frac{3}{2}}\arctan\left[(2-a)(4a-a^2)^{-\frac{1}{2}}\right]\,da\\
&=\arctan\left[(2-a)(4a-a^2)^{-\frac{1}{2}}\right]\left(\frac{(2-a)(4a-a^2)^{\frac{3}{2}}}{4}+\frac{3(2-a)(4a-a^2)^{\frac{1}{2}}}{2}+6\arcsin\left[\frac{2-a}{2}\right]\right)\\
&\quad-\frac{(2-a)^2}{2}+\frac{(2-a)^4}{16}-\frac{3(2-a)^2}{4}-3\left(\arcsin\left[\frac{2-a}{2}\right]\right)^2
\end{aligned}\tag{22}$$

Plug (22) into (20),

$$2\int_0^2\int_0^{\frac{1}{6}(4a-a^2)^{\frac{3}{2}}}\arctan((2-a)6^{-\frac{1}{3}}u^{-\frac{1}{3}})\,du\,da=\frac{1}{4}(-4+\pi^2)$$

□

Lemma 4.4.

$$\int_0^2\int_{\frac{a^2}{4}}^{a}(2-a)\log[1+b-a]\,db\,da=-\frac{1}{2}$$

Note: We will left the calculation of this integral, you can check it with Mathematica.

Lemma 4.5.

$$\int_0^2\int_a^{\frac{a^2+4}{4}}\sqrt{4b-a^2}\arctan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right]\,db\,da=\frac{16}{9}-\frac{\pi^2}{8}$$

Proof. We first integrate $\sqrt{4b-a^2}\arctan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right]$ over b and we can integrate it by using the method of integration by part. Let $u = \arctan\left(\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right)$ and $dv = 4\sqrt{4b-a^2}\,db$. Then

$$\int_a^{\frac{a^2+4}{4}} \sqrt{4b-a^2}\arctan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right] db = -\frac{1}{6}(4a-a^2)^{\frac{3}{2}}\arctan\left[\frac{2-a}{\sqrt{4a-a^2}}\right] + \frac{1}{12}\int_a^{\frac{a^2+4}{4}} 4\sqrt{\frac{(4b-a)^2}{4-4b+a^2}}\,db \tag{23}$$

In order to evaluate the integral in (23), substitute $s = \sqrt{4-4b+a^2}$, $ds = -\frac{2}{\sqrt{4-4b-a^2}}\,db$, so

$$\int_a^{\frac{a^2+4}{4}} 4\sqrt{\frac{(4b-a)^2}{4-4b+a^2}}\,db = -2\int_{2-a}^{0} 4-s^2\,ds = 16-8a-\frac{2}{3}(2-a)^3$$

Then equation (23) is given by

$$\int_a^{\frac{a^2+4}{4}} \sqrt{4b-a^2}\arctan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right] db = -\frac{1}{6}(4a-a^2)^{\frac{3}{2}}\arctan\left[\frac{2-a}{\sqrt{4a-a^2}}\right] + \frac{4}{3}-\frac{2a}{3}-\frac{1}{18}(2-a)^3 \tag{24}$$

Now we can find the whole integral by using formula (24).

$$\int_0^2\int_a^{\frac{a^2+4}{4}} \sqrt{4b-a^2}\arctan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right] db\,da = \int_0^2 -\frac{1}{6}(4a-a^2)^{\frac{3}{2}}\arctan\left[\frac{2-a}{\sqrt{4a-a^2}}\right] da + \frac{10}{9} \tag{25}$$

To evaluate the integral in (25), let $u = \arctan\left[\frac{2-a}{\sqrt{4a-a^2}}\right]$ and $dg = -(4a-a^2)^{\frac{3}{2}}\,da$. Then by using the method of integral by part,

$$-\frac{1}{6}\int_0^2 (4a-a^2)^{\frac{3}{2}}\arctan\left[\frac{2-a}{\sqrt{4a-a^2}}\right] da = \frac{2}{3}-\frac{\pi^2}{8} \tag{26}$$

Now substitute back for (26), (25) becomes

$$\int_0^2\int_a^{\frac{a^2+4}{4}} \sqrt{4b-a^2}\arctan\left[\sqrt{\frac{4+a^2-4b}{4b-a^2}}\right] db\,da = \frac{2}{3}-\frac{\pi^2}{8}+\frac{10}{9} = \frac{16}{9}-\frac{\pi^2}{8}$$

□

References

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