# The probability that a polynomial with integer coefficients has all real roots

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# Problem

What is the probability that a polynomial with random integer coefficients has all real roots?

How to choose a random integer?

- Choose uniformly from $\{-k, -k+1, \dots, k\}$.
- Let $k \to \infty$.

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# Random Linear Polynomial

$p(x) = A_{1,k}x + A_{0,k}$,
$A_{1,k}, A_{0,k}$ random integers in $[-k, k]$

**Question:** what is the probability that $p(x)$ has all real roots?
**Answer:**

$$\text{Solve } p(x) = 0$$

$$x = -\frac{A_{0,k}}{A_{1,k}} \text{ is real } \iff A_{1,k} \neq 0$$

$$P_k = \frac{2k(2k+1)}{(2k+1)^2}$$

$$P = \lim_{k\to\infty} P_k = 1$$

# Random Quadratic Polynomial

$p(x) = A_{2,k}x^2 + A_{1,k}x + A_{0,k}$,
$A_{0,k}, A_{1,k}, A_{2,k}$ random integers in $[-k, k]$

**Question:** what is the probability that $p(x)$ has all real roots?
**Answer:**

$$\text{Solve } p(x) = 0$$
$$x \text{ is real} \iff A_{1,k}^2 - 4A_{0,k}A_{2,k} \geq 0$$
$$P = ?$$

# Solution:

Define F on $\mathbb{R}^3$:
$F(A, B, C) = 1$ if $A^2 - 4BC > 0$,
$F(A, B, C) = 0$ elsewhere.

$$N(k,3) = \sum_{A=-k}^{k} \sum_{B=-k}^{k} \sum_{C=-k}^{k} F(A,B,C)$$
$$= k^3 \sum_{A=-k}^{k} \sum_{B=-k}^{k} \sum_{C=-k}^{k} F\left(\frac{A}{k}, \frac{B}{k}, \frac{C}{k}\right)\Delta\left(\frac{C}{k}\right)\Delta\left(\frac{B}{k}\right)\Delta\left(\frac{A}{k}\right)$$

where, for example, $\Delta(\frac{A}{k})$ represents the change in $\frac{A}{k}$, that is, $\frac{1}{k}$.

# Proof

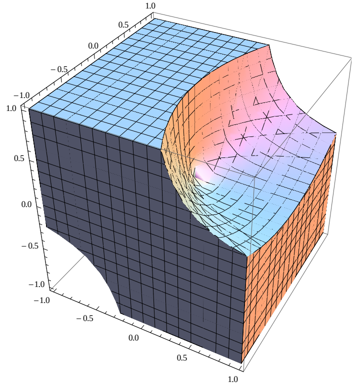
Since $T(k,3) = (2k+1)^3$, thus

$$\lim_{k\to\infty} \frac{N(k,3)}{T(k,3)}$$
$$= \lim_{k\to\infty} \frac{k^3}{(2k+1)^3} \sum_{A=-k}^{k} \sum_{B=-k}^{k} \sum_{C=-k}^{k} F(\frac{A}{k}, \frac{B}{k}, \frac{C}{k})\Delta(\frac{C}{k})\Delta(\frac{B}{k})\Delta(\frac{A}{k})$$
$$= \frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} F(A,B,C)\,dC\,dB\,dA \text{ since F is Riemann integrable}$$
$$= \frac{1}{72}(41 + 3\log 4)$$

# Proof

Therefore, we have

$$P(3) = \frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} F(A,B,C)\,dC\,dB\,dA = \frac{1}{72}(41 + 3\log 4)$$



# Theorem

**Discrete:**

$$f_{k,n}(x) = A_{n,k}x^n + A_{n-1,k}x^{n-1} + \ldots + A_{1,k}x + A_{0,k}$$

$A_{j,k}$ random integers in $\{-k, -k+1, \ldots, k-1, k\}$

$$P(k,n) = \frac{N(k,n)}{T(k,n)}$$

**Continuous:**

$$f_n(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$$

$A_k \in [-1, 1]$ uniformly random

$P(n)$

**Theorem**

$\lim_{k\to\infty} P(k,n) = P(n)$

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## Theorem

$\lim_{k\to\infty} P(k,n) = P(n)$

# General Approach

Discrete problem $\rightarrow$ Continuous problem

Riemann Sum works when F is "almost" continuous

F is continuous except on a set of measure zero

# General Approach

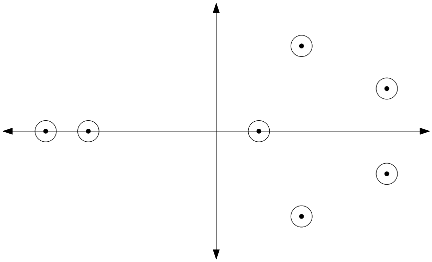
$$f(x) = A_n x^n + \ldots + A_1 x + A_0$$

$$F(A_n, \ldots, A_1, A_0) = \begin{cases} 1 & \text{if } f \text{ has all real roots} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Discriminant } D(f) = \prod_{i<j} (r_i - r_j)^2 = p(A_n, A_{n-1}, \ldots, A_0)$$

$p(A_n, A_{n-1}, \ldots, A_0) = 0$ on a set of measure zero.

# Rouche's Theorem



**Theorem (Rouche's Theorem)**

*If $|f(z) - g(z)| \leq |f(z)|$ for $z$ on a contour $\gamma$,*
*then $f$ and $g$ have the same number of zeros inside $\gamma$.*

Small change in coefficients $\Rightarrow$ small change in roots.

# Result

Since $F$ is almost continuous except at points where $f(x)$ has double roots, thus

$$\lim_{k\to\infty}(P(A_{n,k}x^n+\ldots+A_{1,k}x+A_{0,k}\text{ has all real roots}))$$
$$=P(A_nx^n+\ldots+A_1x+A_0\text{ has all real roots})$$

# Application

$p(x) = A_{3,k}x^3 + A_{2,k}x^2 + A_{1,k}x + A_{0,k}$,
$A_{j,k}$ $(0 \leq j \leq 3)$ random integers in $[-k, k]$

**Question:** what is the probability that $p(x)$ has all real roots?
**Answer:** Cubic Discriminant
$D(A_3, A_2, A_1, A_0) = A_1^2 A_2^2 - 4A_0 A_2^3 - 4A_1^3 A_3 + 18A_0 A_1 A_2 A_3 - 27A_0^2 A_3^2$

$$F(A_3, A_2, A_1, A_0) = \begin{cases} 1 & \text{if } D \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{k \to \infty} P(k, n) = P(n) = \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} F \, dA_3 \, dA_2 \, dA_1 \, dA_0$$
$$\approx 0.2219$$

# Question

Questions?