Functional limit laws for recurrent excited random walks

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Joint work with Elena Kosygina

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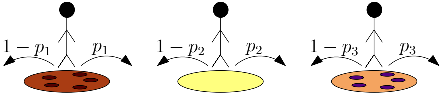
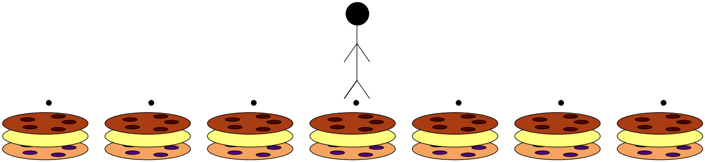
Excited (Cookie) Random Walks

Cookie Environment

- M cookies at each site.
- Cookie *strengths* $p_1, p_2, \ldots, p_M \in (0, 1)$.
- Eating a cookie induces a drift for the next step.





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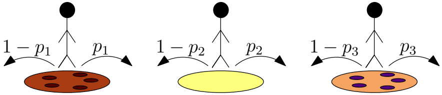

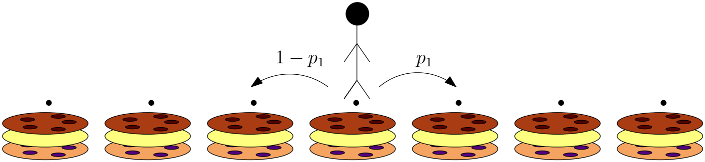

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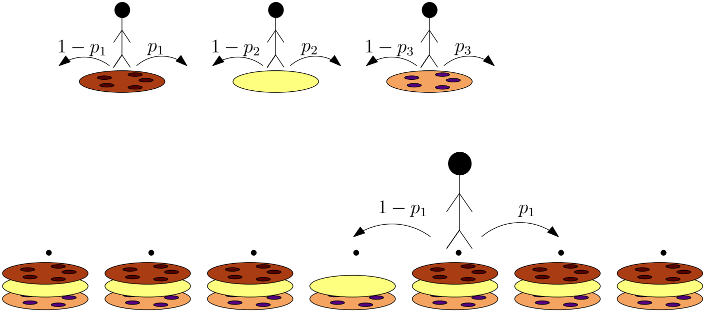

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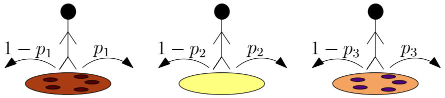

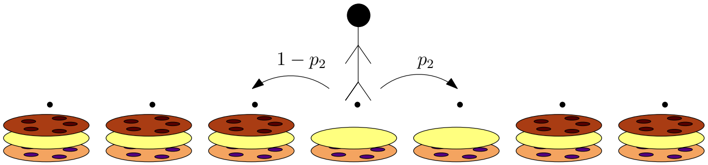

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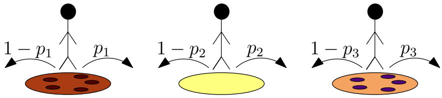

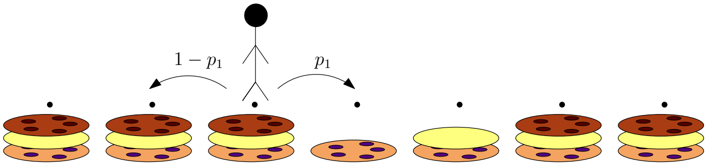

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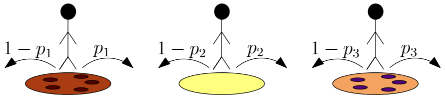

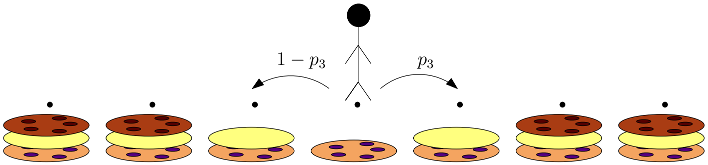

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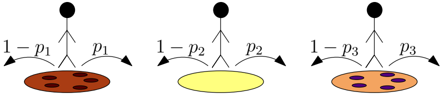

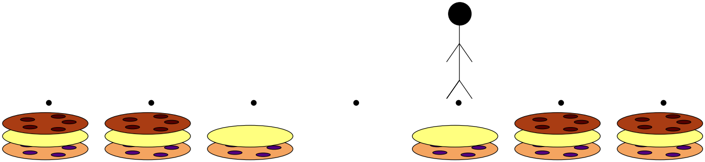

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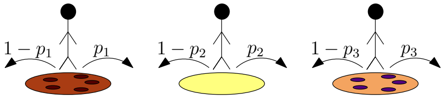

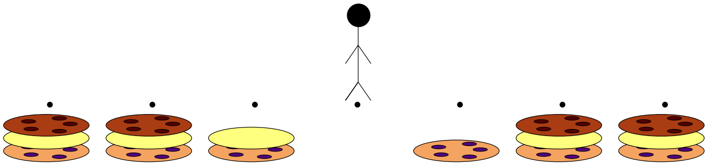

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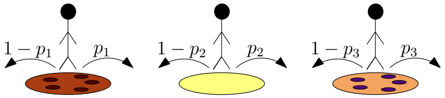

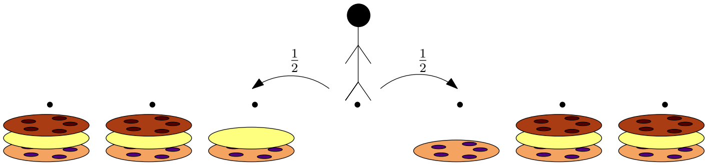

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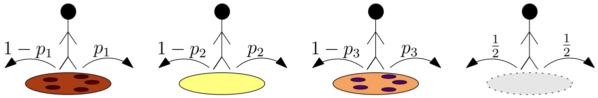
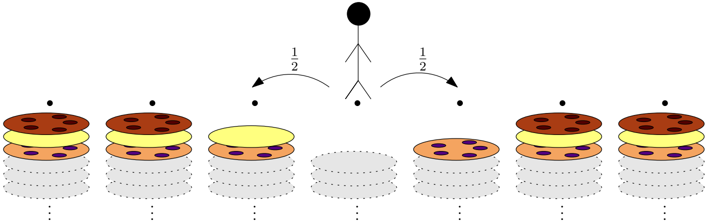
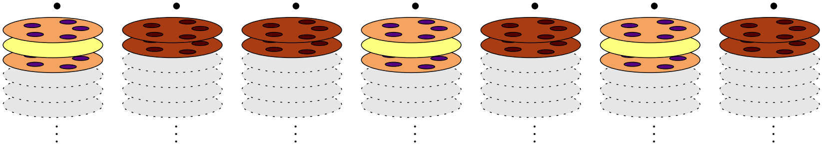




Excited (Cookie) Random Walks

Random i.i.d. cookie environments

- $\omega_x(j)$ – strength of j-th cookie at site x.
- Cookie environment $\omega = \{\omega_x\}$ is i.i.d.
  Cookies *within* a stack may be dependent.



Recurrence/Transience and LLN

Average drift per site

$$\delta = E\left[\sum_{j=1}^{M}(2\omega_0(j) - 1)\right]$$

(deterministic cookie environments: $\delta = \sum_{j=1}^{M}(2p_j - 1)$)

Theorem (Zerner '05, Zerner & Kosygina '08)

The cookie RW is recurrent if and only if $\delta \in [-1, 1]$.

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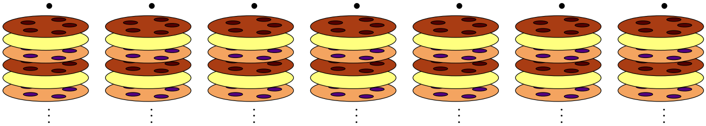
Theorem (Basdevant & Singh '07, Zerner & Kosygina '08)

$\lim_{n\to\infty} X_n/n = v_0$, *and* $v_0 > 0 \iff \delta > 2$.

No explicit formula is known for v_0.

Periodic cookie stacks

- Periodic cookie sequence $p_1, p_2, \ldots, p_M, p_1, p_2, \ldots$
- Average value $\bar{p} = \frac{1}{M} \sum_{j=1}^{M} p_j = \frac{1}{2}$.



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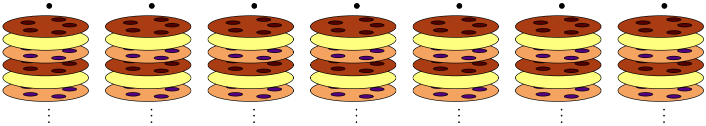

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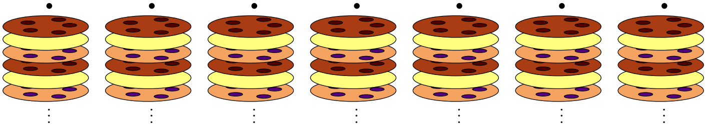

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Note: $\lim_{n\to\infty} \sum_{j=1}^{n} (2p_j - 1)$ doesn't exist.

Periodic cookie stacks - recurrence/transience

$$\theta = \frac{\sum_{j=1}^M \sum_{i=1}^j (1-p_j)(2p_i - 1)}{2\sum_{j=1}^M p_j(1-p_j)} \quad \text{and} \quad \tilde{\theta} = \frac{\sum_{j=1}^M \sum_{i=1}^j p_j(1-2p_i)}{2\sum_{j=1}^M p_j(1-p_j)}.$$

Theorem (Kozma, Shinkar, Orenshtein)

For excited random walks with periodic cookie stacks and $\bar{p} = \frac{1}{2}$,

- $\theta > 1$ *implies* $X_n \to +\infty$.
- $\tilde{\theta} > 1$ *implies* $X_n \to -\infty$.
- $\max\{\theta, \tilde{\theta}\} \leq 1$ *implies* X_n *is recurrent.*

Note: $\theta + \tilde{\theta} < 1$.

Periodic cookie stacks - Example

$$\underbrace{p,\dots,p}_{M/2},\underbrace{1-p,\dots,1-p}_{M/2},\underbrace{p,\dots,p}_{M/2},\underbrace{1-p,\dots,1-p}_{M/2},\dots$$

parameters

$$\theta = \frac{(\frac{M}{2}-(2p-1))(2p-1)}{8p(1-p)}$$

$$\tilde{\theta} = \frac{(\frac{M}{2}+(2p-1))(1-2p)}{8p(1-p)}$$

Periodic cookie stacks - LLN

Limiting speed exists (Kosygina-Zerner '13)

$$v_0 = \lim_{n\to\infty} \frac{X_n}{n}.$$

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Theorem (Kosygina & P. '15)

For excited random walks with periodic cookie stacks and $\bar{p} = \frac{1}{2}$,

- *If $\theta > 2$ then $v_0 > 0$.*
- *If $\tilde{\theta} > 2$ then $v_0 < 0$.*
- *If $\max\{\theta, \tilde{\theta}\} \leq 2$ then $v_0 = 0$.*

Periodic cookie stacks - limiting distributions

Limiting distributions when transient to the right ($\theta > 1$).

Theorem (Kosygina & P. '15)

For excited random walks with periodic cookie stacks and $\bar{p} = \frac{1}{2}$, the following limiting distributions hold

Regime	***Re-scaling***	***Limiting Distribution***
$\theta \in (1,2)$	$\frac{X_n}{n^{\theta/2}}$	$\left(\frac{\theta}{2}\text{-stable}\right)^{-\theta/2}$
$\theta \in (2,4)$	$\frac{X_n - nv_0}{n^{2/\theta}}$	*Totally asymmetric* $\frac{\theta}{2}$*-stable*
$\theta > 4$	$\frac{X_n - nv_0}{A\sqrt{n}}$	*Gaussian*

Results are also known for $\theta = 1$ *and* $\theta = 2$.

Similar limiting distributions for ERW with *M* cookies per site.

Recurrent ERW - scaling limits

What are the scaling limits for recurrent ERW?

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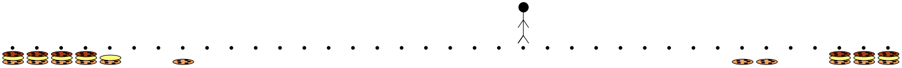
Case I: M cookies per site



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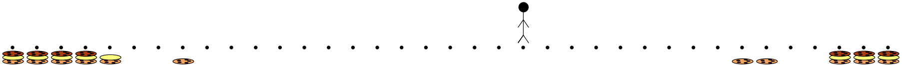

- Like Brownian motion in interior of range
- Gets additional drift at boundary of range

Perturbed Brownian Motion

(α, β)-perturbed Brownian motion

$$Z_t = B_t + \alpha \sup_{s \le t} Z_s + \beta \inf_{s \le t} Z_s.$$

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- Defined for $\alpha, \beta < 1$
- Pathwise unique continuous solution (Perman, Werner ’97 & Chaumont, Doney ’99)

Recurrent ERW - scaling limits

Case I: M cookies per site. $|\delta| \leq 1$.

Theorem (Dolgopyat & Kosygina '12)

- ***Non-boundary case:*** $\delta \in (-1, 1)$.
  $\left\{ \frac{X_{nt}}{\sqrt{n}} \right\}_{t \geq 0}$ *converges to a* $(\delta, -\delta)$*-perturbed Brownian motion.*

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  There exists an $a \in (0, \infty)$ *so that* $\left\{ \frac{X_{nt}}{a\sqrt{n}\log n} \right\}_{t \geq 0}$ *converges to the running supremum of a standard Brownian motion* $B_t^* = \sup_{s \leq t} B_s$.

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Are there similar scaling limits for periodic cookie stacks?

Recurrent ERW - scaling limits

Case II: periodic cookie stacks. $\theta, \tilde{\theta} \leq 1$.

Theorem (Kosygina & Peterson '16)

- ***Non-boundary case:*** $\theta, \tilde{\theta} < 1$.
  $\left\{ \frac{X_{nt}}{a\sqrt{n}} \right\}_{t \geq 0}$ *converges to a* $(\theta, \tilde{\theta})$*-perturbed Brownian motion, with* $a = \frac{1}{2} \left(\frac{1}{M} \sum_{j=1}^{M} p_j(1 - p_j) \right)^{-1/2}$.
- ***Boundary Case:*** $\theta = 1$.
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