Weak quenched limiting distributions of a one-dimensional random walk in a random environment

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RWRE in $\mathbb{Z}$ with i.i.d. environment

An *environment* $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in \Omega = [0,1]^{\mathbb{Z}}$.
P an i.i.d. product measure on Ω.

$$x-1 \xleftarrow{\;1-\omega_x\;} x \xrightarrow{\;\omega_x\;} x+1$$

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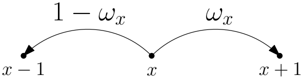

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X_n a random walk: $X_0 = 0$, and

$$P_\omega(X_{n+1} = y+1 | X_n = y) := \omega_y$$

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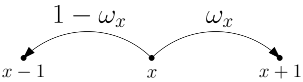

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Averaged law $\mathbb{P}$: average over environments.

$$\mathbb{P}(G) := \int_\Omega P_\omega(G)dP(\omega)$$

Transience Criterion

A crucial statistic is:

$$\rho_x := \frac{1-\omega_x}{\omega_x}$$

Theorem (Solomon '75)

1. *If* $E_P(\log \rho_0) < 0$ *then,* $\lim_{n\to\infty} X_n = +\infty$, $\mathbb{P} - a.s.$
2. *If* $E_P(\log \rho_0) > 0$ *then,* $\lim_{n\to\infty} X_n = -\infty$, $\mathbb{P} - a.s.$
3. *If* $E_P(\log \rho_0) = 0$ *then,* X_n *is recurrent.*

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3. *If* $E_P(\log \rho_0) = 0$ *then,* X_n *is recurrent.*

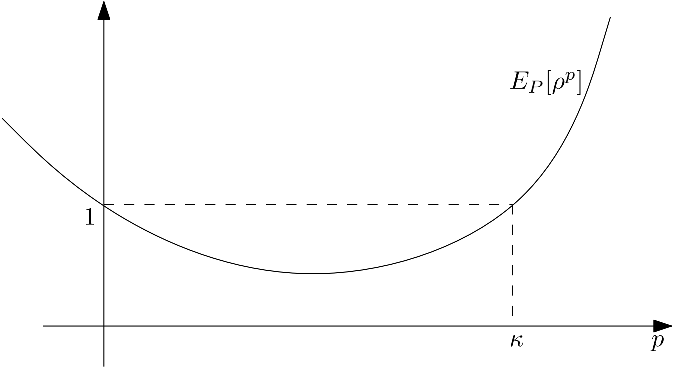
Moreover,

$$\lim_{n\to\infty} \frac{X_n}{n} = v_P > 0 \iff E_P \rho_0 < 1.$$

Scale Parameter $\kappa(P)$ for Transient RWRE

Assume $E_P(\log \rho) < 0$ *(transience to the right)*.
Define $\kappa = \kappa(P)$ by

$$E_P \rho^\kappa = 1.$$



κ is related to the strength of the "traps".

Averaged limiting distribution

Theorem (Kesten, Kozlov, Spitzer '75)

There exists a constant b such that

$$(a) \quad \kappa \in (0,1) \Rightarrow \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n}{n^\kappa} \leq x \right) = 1 - L_{\kappa,b}(x^{-1/\kappa})$$

$$(b) \quad \kappa \in (1,2) \Rightarrow \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - nv_P}{n^{1/\kappa}} \leq x \right) = 1 - L_{\kappa,b}(-x)$$

$$(c) \quad \kappa > 2 \Rightarrow \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - nv_P}{b\sqrt{n}} \leq x \right) = \Phi(x)$$

where $L_{\kappa,b}$ is an κ-stable distribution function.

Remark: Also limiting distributions for $\kappa = 1, 2$ with log corrections to centering or scaling.

Averaged Limiting Distribution (Hitting Times)

$$T_n := \inf\{k \geq 0 : X_k = n\}$$

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Totally Asymmetric κ-Stable Distribution

Characteristic Function of $L_{\kappa,b}$:

$$\exp\left\{-b|t|^\kappa\left(1 - i\frac{t}{|t|}\tan(\pi\kappa/2)\right)\right\}$$

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If $Y \geq 0$ and

$$P(Y_1 > x) \sim Cx^{-\kappa}, \qquad \text{as } x \to \infty,$$

then Y is in the domain of attraction of $L_{\kappa,b}$.

$$T_n = \sum_{i=1}^{n}(T_i - T_{i-1})$$

Quenched Limiting Distribution: Gaussian Regime

Theorem (Goldsheid '06, P. '06)

If $\kappa > 2$ then

$$\lim_{n\to\infty} P_\omega\left(\frac{T_n - E_\omega T_n}{\sigma\sqrt{n}} \le x\right) = \Phi(x), \quad P - a.s.$$

where $\sigma^2 = E_P(Var_\omega T_1)$, and

$$\lim_{n\to\infty} P_\omega\left(\frac{X_n - nv_P + Z_n(\omega)}{v_P^{3/2}\sigma\sqrt{n}} \le x\right) = \Phi(x), \quad P - a.s.$$

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where $Z_n(\omega)$ depends only on the environment.

Cannot remove correction term $Z_n(\omega)$.
Scaling is different from averaged CLT.

Quenched Limiting Distribution: Gaussian Regime

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Quenched CLT $\Rightarrow$ Averaged CLT:

$$\frac{T_n - n/v_P}{\sqrt{n}} = \frac{T_n - E_\omega T_n}{\sqrt{n}} - \frac{E_\omega T_n - n/v_P}{\sqrt{n}}$$

- Terms on right are asymptotically independent.
- $(E_\omega T_n - n/v_P)/\sqrt{n}$ is approximately mean zero Gaussian.

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Question

What happens when $\kappa < 2$?
Do we get quenched stable laws?

Quenched Limits: $\kappa < 2$

Theorem (P.'07)

If $\kappa < 2$ then $P-$ a.s. there exist random subsequences $n_k = n_k(\omega)$, and $m_k = m_k(\omega)$ such that

$$(a) \quad \lim_{k\to\infty} P_\omega\left(\frac{T_{n_k} - E_\omega T_{n_k}}{\sqrt{Var_\omega T_{n_k}}} \le x \right) = \Phi(x)$$

$$(b) \quad \lim_{k\to\infty} P_\omega\left(\frac{T_{m_k} - E_\omega T_{m_k}}{\sqrt{Var_\omega T_{m_k}}} \le x \right) = \begin{cases} 0 & \text{if } x < -1 \\ 1 - e^{-x-1} & \text{if } x \ge -1 \end{cases}$$

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FAQ

- Which limit is more likely?

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- Which limit is more likely?
- What other subsequential limits are possible?

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FAQ

- Which limit is more likely?
- What other subsequential limits are possible?
- Where do the κ-stable distributions come from.

Weak Quenched Limits: $\kappa < 2$

For any $x \in \mathbb{R}$,

$$P_\omega\left(\frac{T_n - E_\omega T_n}{n^{1/\kappa}} \le x \right)$$

is a random variable.

- Doesn't converge almost surely to a deterministic constant.
- Maybe converges in some weaker sense (in distribution).

Random distributions

$\mathcal{M}_1 =$ Probability distributions on $\mathbb{R}$.
- Equip with topology of convergence in distribution.

Quenched distributions are $\mathcal{M}_1$-valued random variables.

Quenched CLT ($\kappa > 2$):

$$\mu_{n,\omega}(A) = P_\omega\left(\frac{T_n - E_\omega T_n}{\sigma\sqrt{n}} \in A \right).$$

Then,

$$\lim_{n\to\infty} \mu_{n,\omega} = \gamma_0 \in \mathcal{M}_1, \quad P - a.s.$$

where $\gamma_0 \in \mathcal{M}_1$ is the standard normal distribution.

Weak Quenched Limiting Distributions: $\kappa < 2$

$$\mu_{n,\omega}(A) = P_\omega\left(\frac{T_n - E_\omega T_n}{n^{1/\kappa}} \in A\right)$$

Theorem (P. & Samorodnitsky '10)

If $\kappa < 2$, then there exists a $\lambda > 0$ such that

$$\mu_{n,\omega} \underset{n\to\infty}{\Longrightarrow} \mu_{\lambda,\kappa},$$

where $\mu_{\lambda,\kappa}$ is a random probability distribution defined by

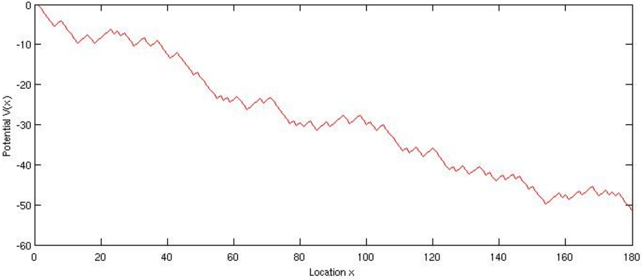
$$\mu_{\lambda,\kappa}(A) = \mathbf{P}\left(\sum_{i=1}^{\infty} a_i(\tau_i - 1) \in A \,\middle|\, a_i,\ i \geq 1\right)$$

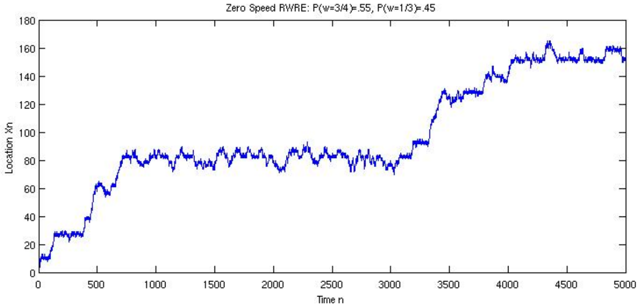
and $N = \sum_{i\geq 1}\delta_{a_i}$ is a non-homogeneous Poisson point process with intensity measure $\lambda\kappa x^{-\kappa-1}\,dx$.

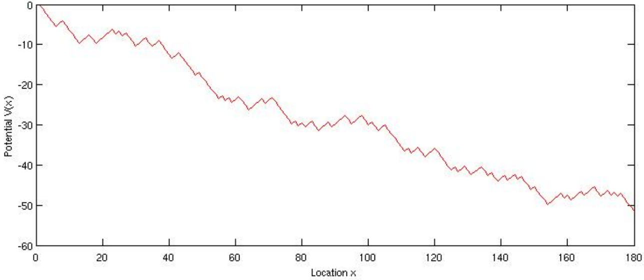
Potential and Traps

$$V(i) := \begin{cases} \sum_{k=0}^{i-1} \log \rho_k, & i > 0 \\ 0, & i = 0 \\ \sum_{k=i}^{-1} -\log \rho_k, & i < 0 \end{cases}$$

Trap: Atypical section where the potential is increasing.



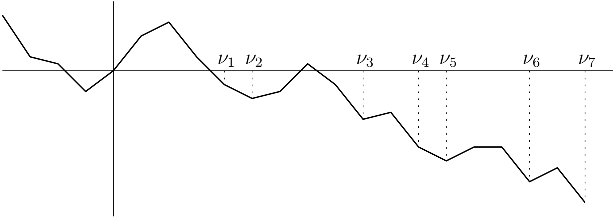

Zero Speed RWRE: P(w=3/4)=.55, P(w=1/3)=.45
Location Xn
Time n
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
0
20
40
60
80
100
120
140
160
180


Potential V(x)
Location x
0
20
40
60
80
100
120
140
160
180
0
-10
-20
-30
-40
-50
-60

Blocks of the environment

Ladder locations $\{\nu_n\}$ defined by $\nu_0 = 0$,

$$\nu_n := \inf\{i > \nu_{n-1} : V(i) < V(\nu_{n-1})\}$$



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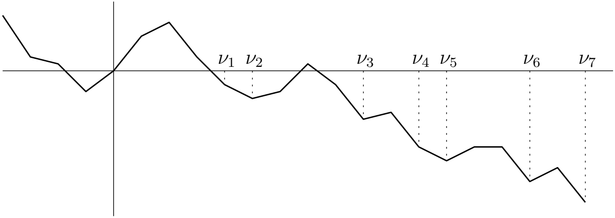

$$Q(\cdot) = P\left(\cdot \,|\{V(i) > 0, \forall i < 0\}\right)$$

Under Q, the environment is stationary under shifts of the ν_i.

Reduction to T_{ν_n}

$\nu_n = \sum_{i=1}^n (\nu_i - \nu_{i-1})$.

LLN implies

$$\frac{\nu_n}{n} \to \bar{\nu} = E_P \nu_1$$

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Enough to study

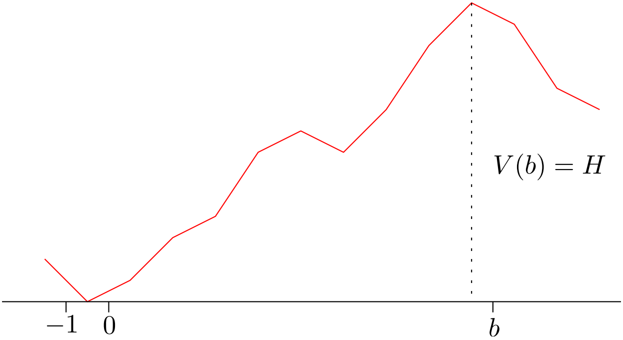
$$\phi_{n,\omega}(\cdot) = P_\omega \left(\frac{T_{\nu_n} - E_\omega T_{\nu_n}}{n^{1/\kappa}} \in \cdot \right)$$

Want to show $\phi_{n,\omega} \Longrightarrow \mu_{\lambda,\kappa}$ for some λ.

Escaping Traps

Probability of escaping a trap of Height H.

$$P_\omega(T_b < T_{-1}) = \frac{1}{1 + \sum_{j=1}^{b} e^{V(j)}}$$



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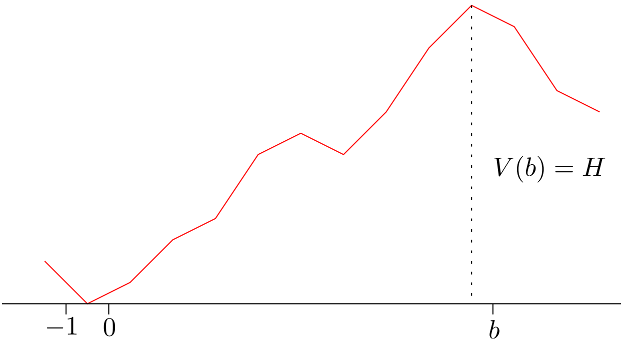
$$P_\omega(T_b < T_{-1}) = \frac{1}{1 + \sum_{j=1}^{b} e^{V(j)}} \approx e^{-V(b)} = e^{-H}$$



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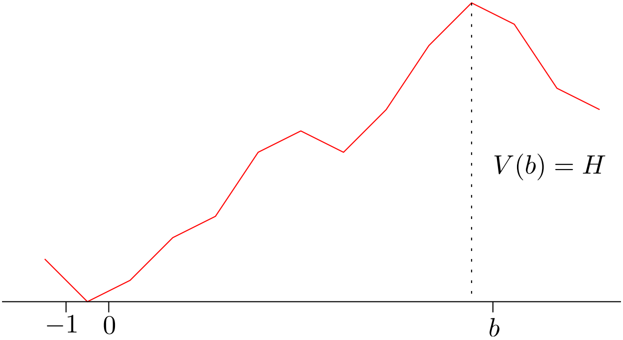
Time to escape trap of height $H \approx Exp(e^{-H})$.



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Want: $T_\nu \overset{Law}{\approx} (E_\omega T_\nu)\tau$.

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Decompose T_ν into trials

$$T_\nu = S + \sum_{i=1}^{G} F_i$$

Where

$$S \sim (T_\nu \mid T_\nu < T_0^+)$$
$$F_i \sim (T_0^+ \mid T_0^+ < T_\nu)$$
$$G \sim \text{Geo}(p_\omega), \qquad \text{where } p_\omega = P_\omega(T_\nu < T_0^+)$$

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$$\begin{aligned} S &\sim (T_\nu \mid T_\nu < T_0^+) \\ F_i &\sim (T_0^+ \mid T_0^+ < T_\nu) \\ G &\sim \text{Geo}(p_\omega), \qquad \text{where } p_\omega = P_\omega(T_\nu < T_0^+) \end{aligned}$$

Couple G with an exponential τ

$$G = \left\lfloor \frac{\tau}{\log(1/(1-p_\omega))} \right\rfloor.$$

Heuristics of Quenched Limit Laws

$$T_{\nu_n} = \sum_{i=1}^{n} (T_{\nu_i} - T_{\nu_{i-1}}) \overset{Law}{\approx} \sum_{i=1}^{n} \beta_i \tau_i$$

where τ_i are i.i.d. Exp(1) random variables and

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Reduced to study of

$$\sigma_{n,\omega}(\cdot) = P_\omega\left(\frac{1}{n^{1/\kappa}} \sum_{i=1}^{n} \beta_i(\tau_i - 1) \in \cdot \right).$$

where τ_i are i.i.d. exponential(1).

Properties of β_i

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$$Q(\beta_1 > x) = Q(E_\omega T_\nu > x) \sim C x^{-\kappa}.$$

Moreover, if $\kappa < 1$ then there exists a constant $b > 0$ such that

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Kesten showed that $P(E_\omega T_1 > x) \sim C' x^{-\kappa}$.

Point Process Convergence

$\mathcal{M}_p =$ Point processes $\sum_{i\geq 1} \delta_{x_i}$ on $(0,\infty)$.

$$N_n = N_n(\omega) = \sum_{i=1}^{n} \delta_{\beta_i/n^{1/\kappa}}$$

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Remark: $N_{\lambda,\kappa} = \sum_{i\geq 1} \delta_{a_i}$ can be constructed by letting

$$a_i = \Gamma_i^{-1/\kappa},$$

where Γ_i are the arrivals of a homogeneous Poisson(λ) point process.

Define $H : \mathcal{M}_p \longrightarrow \mathcal{M}_1$ by

$$\zeta = \sum_{i \geq 1} \delta_{x_i} \Longrightarrow H(\zeta)(A) = \mathbf{P}\left(\sum_{i\geq 1} x_i(\tau_i - 1) \in A \middle| x_i,\, i \geq 1 \right).$$

Remark: Only defined if $\sum x_i^2 < \infty$.

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Recall

- $\sigma_{n,\omega} = H(N_n)$
- $\mu_{\lambda,\kappa} \stackrel{\mathcal{D}}{=} H(N_{\lambda,\kappa})$

Since $N_n \Longrightarrow N_{\lambda,\kappa}$, would like to conclude that

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H is NOT continuous.

For $\varepsilon > 0$ let $H_\varepsilon : \mathcal{M}_p \longrightarrow \mathcal{M}_1$ be defined by

$$\zeta = \sum_{i \geq 1} \delta_{x_i} \Longrightarrow H_\varepsilon(\zeta)(A) = \mathbf{P}\left(\sum_{i \geq 1} x_i \mathbf{1}_{\{x_i > \varepsilon\}}(\tau_i - 1) \in A \,\middle|\, x_i,\ i \geq 1 \right).$$

Claim: H_ε is continuous on

$$\{\zeta \in \mathcal{M}_p : \zeta(\{\varepsilon\}) = 0\} \subset \mathcal{M}_p.$$

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- Let $\zeta_n \to \zeta$ with $\zeta(\{\varepsilon\}) = 0$.
- $\zeta\big|_{[\varepsilon,\infty)} = \sum_{i=1}^M \delta_{x_i}$.
- For n large $\zeta_n\big|_{[\varepsilon,\infty)} = \sum_{i=1}^M \delta_{x_i^{(n)}}$, and $x_i^{(n)} \to x_i$ for all $i \leq M$.

Since $\mathbf{P}(N(\{\varepsilon\}) = 0) = 1$,

$$H_\varepsilon(N_n) \Longrightarrow H_\varepsilon(N).$$

It remains to show

$$\lim_{\varepsilon \to 0} \rho(H_\varepsilon(N), H(N)) = 0, \quad \mathbf{P} - a.s. \tag{1}$$

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} Q\left(\rho(H_\varepsilon(N_n), H(N_n)) \geq \delta\right) = 0. \tag{2}$$

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where ρ is the Prohorov metric on $\mathcal{M}_1$.

$$\rho(\mu, \pi) = \inf\left\{\delta \,:\, \mu(A) \leq \pi(A^{(\delta)}) + \delta, \text{ for all Borel } A\right\}$$

$H_\varepsilon(N) \to H(N)$

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\begin{aligned}
H_\varepsilon(N)(A) &= \mathbf{P}\left(\sum_{i\geq 1} a_i \mathbf{1}_{\{a_i > \varepsilon\}} (\tau_i - 1) \in A \right) \\
&\leq \mathbf{P}\left(\sum_{i\geq 1} a_i (\tau_i - 1) \in A^{(\delta)} \right) + \mathbf{P}\left(\left| \sum_{i\geq 1} a_i \mathbf{1}_{\{a_i \leq \varepsilon\}} (\tau_i - 1) \right| \geq \delta \right) \\
&\leq H(N)(A^{(\delta)}) + \frac{1}{\delta^2} \sum_{i\geq 1} a_i^2 \mathbf{1}_{\{a_i \leq \varepsilon\}}
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Thus

$$
\rho(H_\varepsilon(N), H(N)) \leq \left(\sum_{i\geq 1} a_i^2 \mathbf{1}_{\{a_i \leq \varepsilon\}} \right)^{1/3}
$$

Similarly

$$Q\left(\rho(H_\varepsilon(N_n), H(N_n)) > \delta\right) \leq Q\left(\frac{1}{n^{2/\kappa}} \sum_{i\geq 1} \beta_i^2 \mathbf{1}_{\{\beta_i/n^{1/\kappa} \leq \varepsilon\}} > \delta^3 \right) \leq \frac{n^{1-2/\kappa}}{\delta^3} E_Q\left[\beta_1^2 \mathbf{1}_{\{\beta_1/n^{1/\kappa}\leq\varepsilon\}} \right]$$

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Control expectation with tail decay:

$$E_Q\left[\beta_1^2 \mathbf{1}_{\{\beta_1/n^{1/\kappa} \leq \varepsilon\}} \right] \sim C' \varepsilon^{1-\kappa/2} n^{2/\kappa - 1}.$$

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Thus,

$$Q\left(\rho(H_\varepsilon(N_n), H(N_n)) > \delta\right) \leq \frac{C'}{\delta^3} \varepsilon^{1-\kappa/2}.$$