# Analysis of the Speed of One-Dimensional Excited Random Walks

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## Excited Random Walk (ERW) Background

Excited random walks (*ERWs*) are a self-interacting random walk in which the future behavior of the walk is influenced by the number of times the walk has previously visited its current site, and thus is non-Markovian. We let $M$ be an integer for the number of *cookies* at each site and $p$ be the strength of the cookies. If a walker visits a site with at least one cookie present, they move right with probability $p > \frac{1}{2}$ and left with probability $1-p$. If a site has no remaining cookies, the walker moves left and right with equal probability for that step.

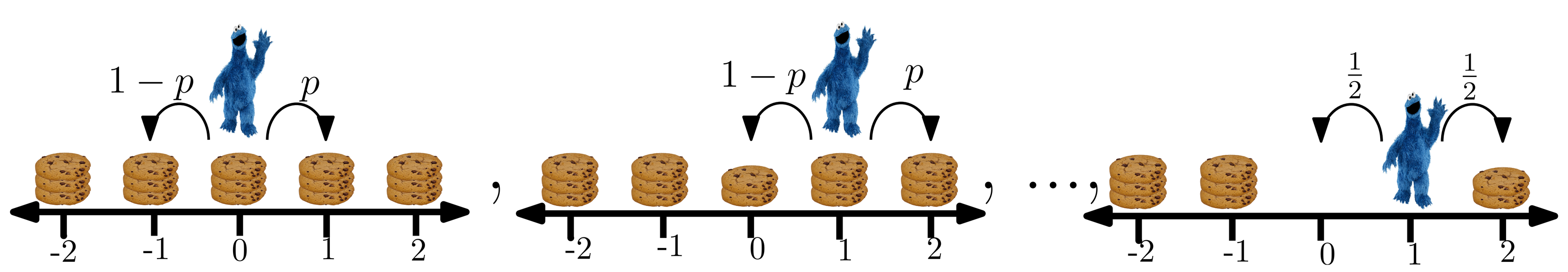



Figure: *The first image in the series shows the cookie environment before the walk begins using 3 cookies of equal strength $p$. Next we have our walk progressed one step, thus one cookie gone, and lastly the walk progressed some steps into the future such that all the cookies have been eaten at the current site.*

### Background Definitions

*Speed* for a random walk is defined as

$$v = V_{M,p} = \lim_{n\to\infty} \frac{X_n}{n}$$

where $X_n$ is the walker's location at time $n$. The *drift*, $\delta$, is defined as

$$\delta = M(2p-1)$$

### Background Theorems

**Theorem [Zerner 2005]**
When $\delta > 1$ we have that $\lim_{n\to\infty} X_n \to +\infty$.
**Theorem [Basdevant and Singh 2008]**
We also have that $V_{M,p} > 0$ if and only if $\delta > 2$.

### Our Goal

Obtain rigorous upper and lower bounds for the speed when $p$ is such that $\delta > 2$ so the speed is positive.

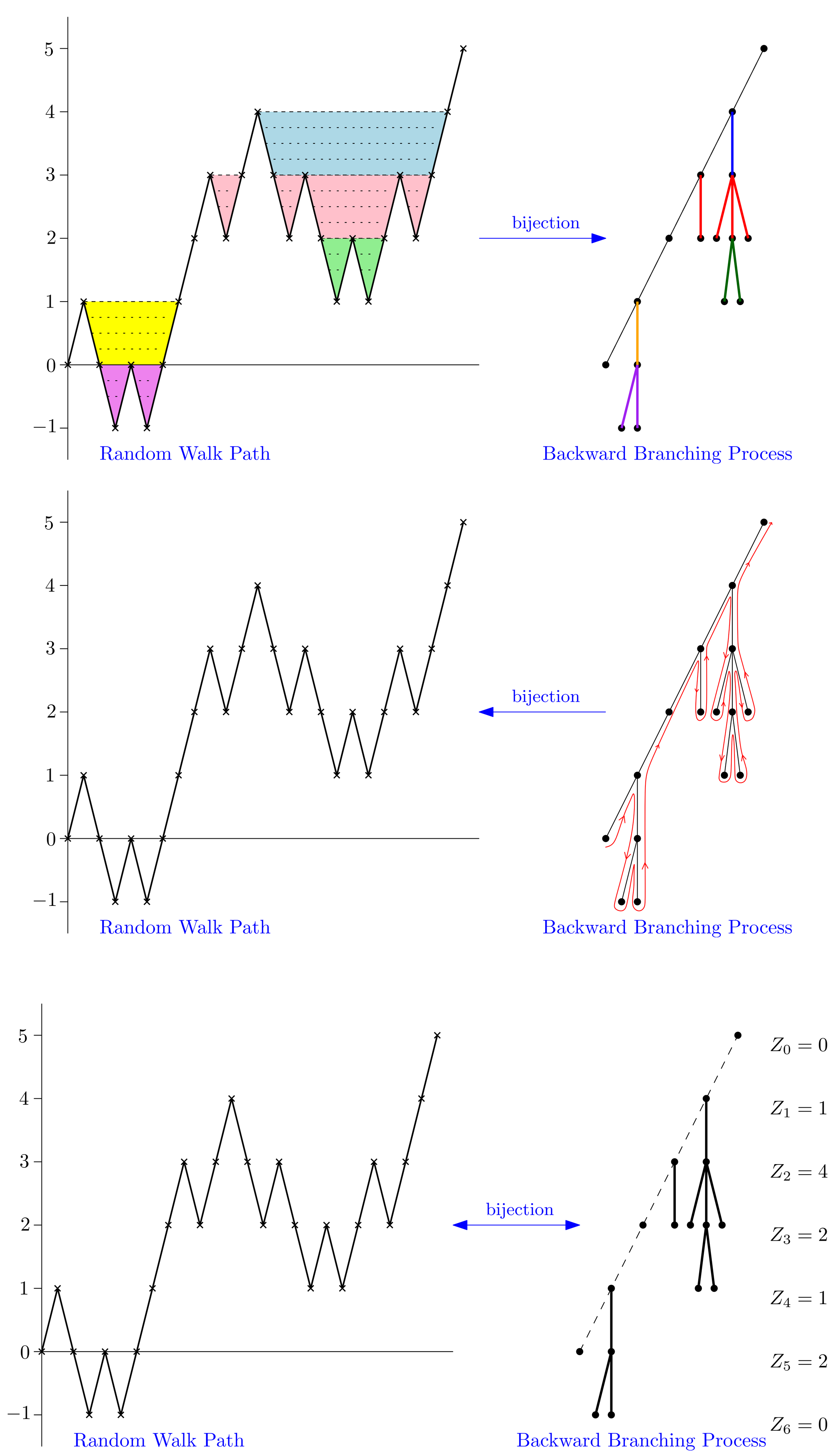
## Backward Branching Process (BBP)

Under appropriate conditions, there is a bijection from ERWs to a Markov chain $Z$ which we call a *backward branching process*.

### Backward Branching Process Theorems

**Theorem [Basdevant and Singh 2008]**
If the ERW is transient then $Z$ is positive recurrent and there exists a unique stationary distribution, $\pi$ and we have that

$$V_{M,p} = \frac{1}{1 + 2E_\pi[Z_0]}$$




## Our Work

We tried many methods, but primarily found success with two. The first was to truncate $Z$ as $Z^{(L)}$ so it had a finite state space of integers $0 \le k \le L$. We then computed the stationary distribution, $\pi^{(L)}$, of each truncation to approximate $\pi$. We also had success extending work by Basdevant and Singh by deriving further relations between values of $\pi$ which explicitly bound the speed.

## Method #1: Truncated Branching process

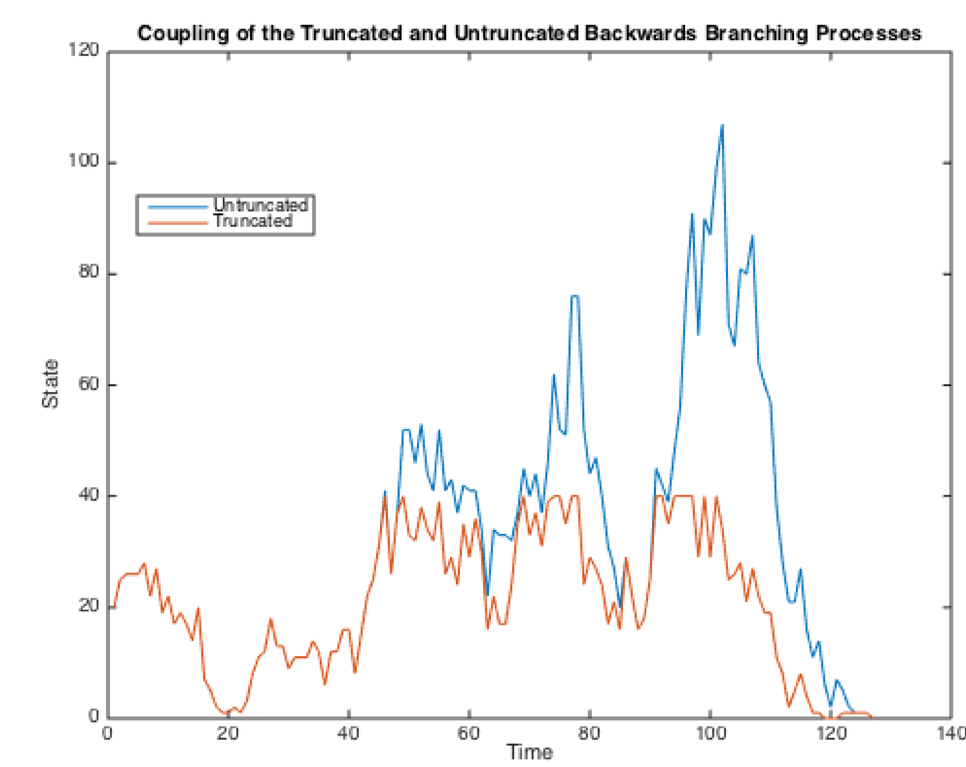



Figure: Here we see how $Z^{(40)}$ varies from $Z$ when coupled so $Z^{(40)} \le Z$.

## Method#1 Results

### Truncation Speed Convergence Theorem

**Theorem 1:** We have that $\lim_{L\to\infty} E_{\pi^{(L)}}[Z_0^{(L)}] = E_\pi[Z_0]$.

In the limit, we have a tight upper bound on the speed. Unfortunately, $\pi^{(L)}$ is unfeasible to compute for large $L$.

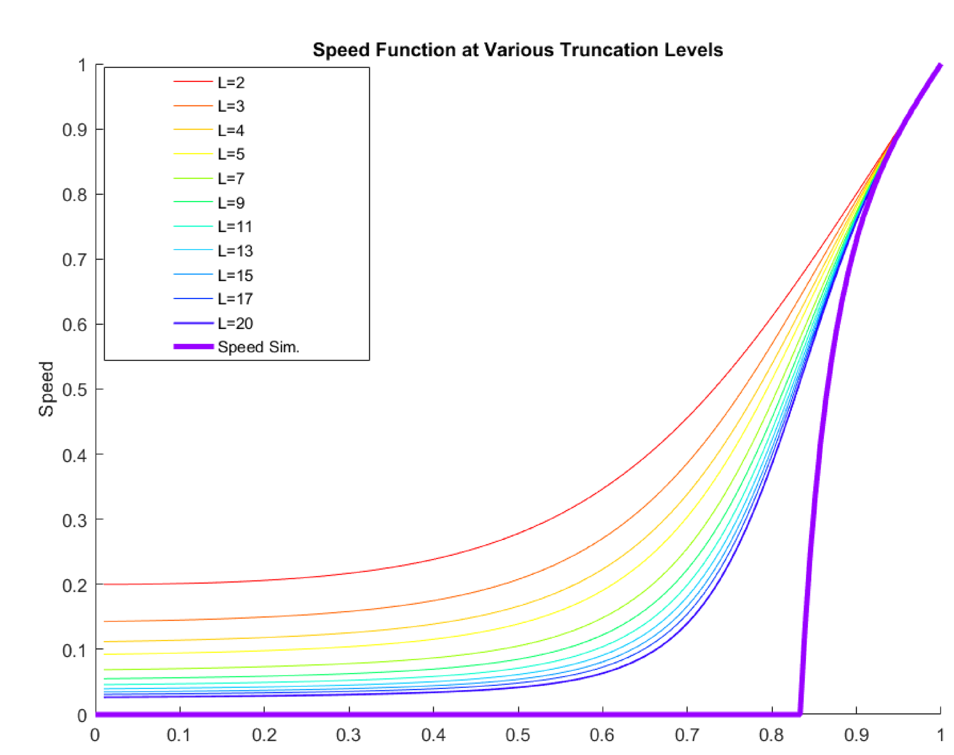



Figure: This graphic shows how the truncated speed approaches the actual speed as $L$ increases with diminishing marginal returns

### Future Directions

**Conjecture:** We believe $V_{M,p}$ is differentiable in $p$.
We'd also like to find better bounds for $M > 3$.

## Acknowledgements

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## Method #2: Bounding Speed via $\pi$

Basdevant and Singh have shown speed can be explicitly calculated in terms of $\pi(0)$ through $\pi(M-2)$. Thus, bounding the speed is equivalent to bounding those values.

## Method #2 Results

### Speed Bounding Theorems

**Theorem 2:** The following linear relation is satisfied by entries in $\pi$:

$$c = \sum_{k=0}^{M-2} a_k \pi(k)$$

where $a_k$ and $c$ are explicitly computable polynomials in $p\ \forall k$. Also, when $M = 3$, we have the following two theorems:
**Theorem 3:** $\pi(0) \ge \pi(0)p(0,0) + \pi(1)p(1,0)$
**Theorem 4:** $\pi(1) \ge \pi(0)p(0,1) + \pi(1)p(1,1)$, where $p(i,j)$ is the transition probability from state $i$ to state $j$ in $Z$

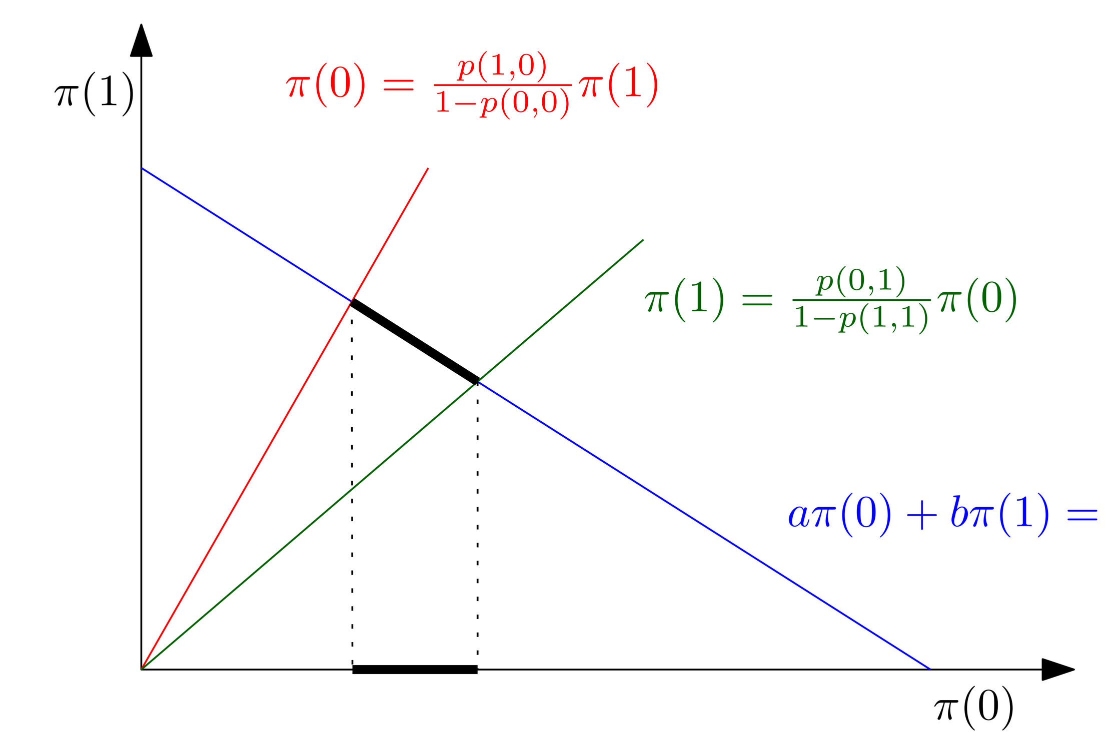
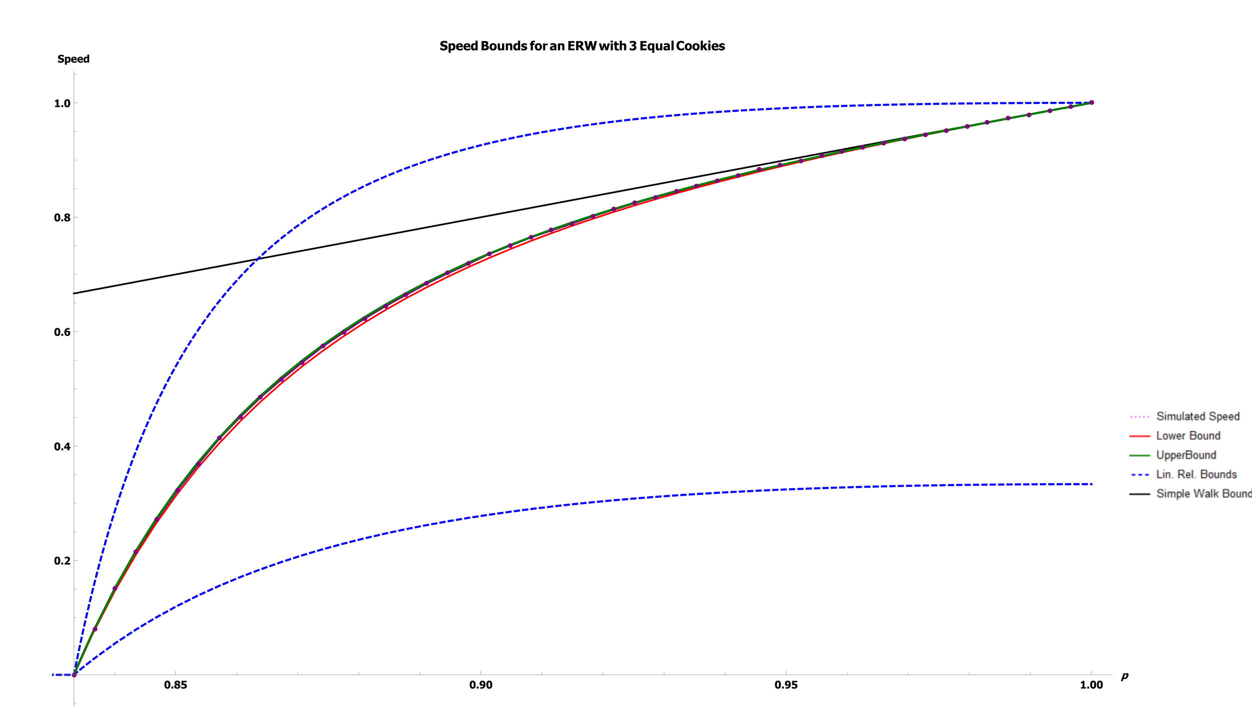



Figure: A qualitative illustration of the bounds we found for $M = 3$

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