Analysis of a Ginzburg-Landau Type Energy Model for Smectic C* Liquid Crystals with Defects

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Abstract. This work investigates properties of a Smectic C* liquid crystal film containing defects that cause distinctive spiral patterns in the film's texture. The phenomena are described by a Ginzburg-Landau type model and the investigation provides a detailed analysis of minimal energy configurations for the film's director field. The study demonstrates the existence of a limiting location for the defects (vortices) so as to minimize a renormalized energy. It is shown that if the degree of the boundary data is positive then the vortices each have degree $+1$ and that they are located away from the boundary. It is proved that the limit of the energies for a sequence of minimizers minus the sum of the energies around their vortices, as the G-L parameter ε tends to zero, is equal to the renormalized energy for the limiting state.

Ce travail étudie les propriétés d'un Smectique C* film de cristaux liquides que contient des défauts qui entraînent des motifs distinctes spirale dans la texture du film. Les phénomès sont décrit par un modéle de type Ginzburg-Landau et la enquête apporte une analyse detaillée des configurations d'énergie minimale pour le champ directeur du film. L'étude démontre l'existence d'un limiter emplacement pour les défauts (tourbillons) pour minimiser une énergie renomrmalisée. Il est montré que si le degré des valeurs limites est positif ensuite les tourbillons chaque ont un degré $+1$ et qu'ils sont situés loin de la frontière. Il est prouvé qu' á la limite des énergies pour un séquence des minimiseurs moins la somme des énergies autour leurs tournillons, comme le paramètre G-L ε tend vers zéro, est égale à la énergie renormalisée pour l'état limitatif.

1. Introduction

We study the occurrence of point defects in a thin ferroelectric smectic C* (Sm C*) liquid crystal by using a director field description based on the Ginzburg-Landau theory. The unknown function u is a vector field in $\mathbb{R}^2$. When convenient, for ease of notation, we view it as a $\mathbb{C}$-valued function such that $u = u^1(x_1,x_2) + iu^2(x_1,x_2)$

for (x_1,x_2) in a bounded domain Ω in $\mathbb{R}^2$. We assume Ω has a smooth (C^3) boundary in the plane and that Ω represents the reference configuration of a very thin liquid crystal material.

We analyze minimizers for a Ginzburg-Landau type energy,

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(k_s(\operatorname{div} u)^2 + k_b(\operatorname{curl} u)^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)dx = \int_\Omega j_\varepsilon(u,\nabla u)\,dx, \tag{1.1}$$

where k_s and k_b represent the two dimensional splay and bend moduli for the film respectively, with $k_s, k_b > 0$. Here $J_\varepsilon(\cdot)$ is defined for $u \in H^1_g(\Omega;\mathbb{R}^2)$, consisting of fields $u \in H^1(\Omega;\mathbb{R}^2)$ with Dirichlet boundary conditions, $u|_{\partial\Omega} = g \in C^3(\partial\Omega;\mathbb{S}^1)$ such that $\deg(g,\partial\Omega) = d > 0$. The variable $\varepsilon > 0$ represents the radius of the defect cores. Previous work has considered the case where $k_s = k_b$, reducing to the classical Ginzburg-Landau functional [1, 2, 3]. Our work focuses on the cases where $k_s \neq k_b$. The elastic energy term of (1.1) is used to model thin film liquid crystals with chirality, such as a Sm C* material. The resulting pattern consists of a family of point defects in the film forming vortices in the molecular texture that spiral in a fashion determined by the relative values for k_s and k_b; see [4].

1.1. Main Results

By denoting $\underline{k} = \min\{k_s, k_b\}$, we can express (1.1) as

$$J_\varepsilon(u) = \bar{J}_\varepsilon(u) + \underline{k}\int_\Omega \det\nabla u\,dx = \bar{J}_\varepsilon(u) + \underline{k}\pi d \tag{1.2}$$

for $u \in H^1_g(\Omega;\mathbb{R}^2)$ where

$$\begin{aligned}\bar{J}_\varepsilon(u) &= \int_\Omega \bar{j}_\varepsilon(u,\nabla u)\,dx \\ &= \begin{cases} \frac{1}{2}\int_\Omega \left(k_s|\nabla u|^2 + (k_b-k_s)(\text{curl } u)^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)dx & \text{if } \underline{k} = k_s \\ \frac{1}{2}\int_\Omega \left((k_b|\nabla u|^2 + (k_s-k_b)(\text{div } u)^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right)dx & \text{if } \underline{k} = k_b. \end{cases}\end{aligned} \tag{1.3}$$

Then $u \in H^1_g(\Omega;\mathbb{R}^2)$ is a minimizer of $J_\varepsilon(u)$ if and only if u is a minimizer for $\bar{J}_\varepsilon(u)$. Hence, it suffices to consider the minimizers of equation (1.3) and analyze this functional. In this way, by the strict convexity of the integral in (1.3), we have the existence of a minimizer u_ε for each ε; see [5].

We need a detailed description of Ω. Let $D \subset \mathbb{R}^2$ be a bounded, simply connected domain with a C^3 boundary Γ_0. For $\ell = 1,\ldots,k$ let $\overline{\Lambda}_\ell \subset D$ be pair-wise disjoint, simply connected sets with C^3 boundaries Γ_ℓ. Consider the domain $\Omega = D \setminus \bigcup_{\ell=1}^{k} \overline{\Lambda}_\ell$ where we take the natural orientation for $\partial\Omega = \bigcup_{\ell=0}^{k} \partial\Gamma_\ell$, such that Γ_0 is oriented counter-clockwise and Γ_ℓ are oriented clockwise for $1 \le \ell \le k$. For each $g \in C^3(\partial\Omega;\mathbb{S}^1)$ set $d_\ell :=$ winding number of $g|_{\Gamma_\ell}$ with respect to the curve's orientation, and denote the degree $d(g,\partial\Omega) := d = \sum_{\ell=0}^{k} d_\ell$. We fix k points, $y_\ell \in \Lambda_\ell$, and set $w(x) = \prod_{\ell=1}^{k}\left(\frac{x-y_\ell}{|x-y_\ell|}\right)^{-d_\ell} = e^{i\zeta(x)}$ for $x \in \overline{\Omega}$. Thus ζ is a multi-valued, harmonic expression such that $\nabla\zeta(x)$ is point-wise well defined. We use $w(x)$ to fix specific representations of functions having boundary values with winding numbers d_ℓ with respect to Γ_ℓ for $1 \le \ell \le k$. The minimizers of the energy functional over H^1_g have a number of structural properties that lead to the first main result of the paper.

Theorem A. *Let $\{u_\varepsilon\}$ be a sequence of minimizers for $J_\varepsilon(u)$ over H^1_g such that $\varepsilon \to 0$. Then there is a subsequence $\{u_{\varepsilon_\ell}\}$, a function $h \in H^1(\Omega)$ and d points $\{a_1, \ldots, a_d\} \in \Omega$ such that*

$$\begin{aligned} &|u_{\varepsilon_\ell}| \to 1 \quad \text{uniformly on compact subsets of } \overline{\Omega} \setminus \{a_1, \ldots, a_d\}, \\ &\text{and more generally } u_{\varepsilon_\ell}(x) \to u_*(x) = e^{i(h(x)+\zeta(x)+\sum_{n=1}^d \theta_{a_n}(x))} \end{aligned} \tag{1.4}$$

in $C^\alpha_{loc}(\overline{\Omega} \setminus \{a_1, \ldots, a_d\})$ and in $C^m_{loc}(\Omega \setminus \{a_1, \ldots, a_d\})$ for every $0 < \alpha < 1$ and integer $m \ge 0$, in which $\theta_{a_n} = \theta_{a_n}(x)$ denotes the polar angle of x with respect to the center a_n.

The d-tuple $\mathbf{a} = (a_1, \ldots, a_d) \in \Omega^d$ represents the point defects within the domain Ω. The energy functional, just as in [1], has a renormalized form,

$$\underline{k}W(\mathbf{b}) + H(\mathbf{b}, k_s, k_b) \quad \text{for } \mathbf{b} \in \Omega^d \tag{1.5}$$

here

$$\begin{aligned} W(\mathbf{b}) =& \frac{1}{2} \int_{\partial\Omega} \Big(2G_{\mathbf{b}}(g \times \partial_\tau g) - (\partial_\nu G_{\mathbf{b}})G_{\mathbf{b}} \Big) d\sigma + \pi d \\ & - \sum_{m \ne n} \pi \ln(|b_n - b_m|) + \sum_{n=1}^d \sum_{\ell=1}^k \pi d_\ell \ln(|b_n - y_\ell|) \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} H(\mathbf{b}, k_s, k_b) &= \inf_\phi \mathscr{H}(\mathbf{b}, \phi, k_s, k_b) \\ &= \begin{cases} \inf_\phi \frac{1}{2} \int_\Omega \Big(k_s |\nabla\phi|^2 + (k_b - k_s)(\operatorname{curl} v)^2 \Big) dx & \text{if } \underline{k} = k_s \\ \inf_\phi \frac{1}{2} \int_\Omega \Big(k_b |\nabla\phi|^2 + (k_s - k_b)(\operatorname{div} v)^2 \Big) dx & \text{if } \underline{k} = k_b \end{cases} \end{aligned} \tag{1.7}$$

where $G_{\mathbf{b}}(x) = \sum_{n=1}^d \ln(|x - b_n|) - \sum_{\ell=1}^k d_\ell \ln(|x - y_\ell|)$, $v(x) = \prod_{n=1}^d \frac{x - b_n}{|x - b_n|} e^{i(\phi(x) + \zeta(x))}$, and $\mathscr{H}(\mathbf{b}, \phi, k_s, k_b)$ is minimized over the class of functions $\phi \in H^1(\Omega)$ such that $v = g$ on $\partial\Omega$. The expression (1.5) is a variant of the renormalized energy from [1]. Moreover the two agree for the case where $k_s = k_b$ and Ω is simply connected. By the definition of (1.5), we have that in order for the renormalized energy to be finite, $b_n \ne b_m$ for $n \ne m$ and $b_n \notin \partial\Omega$ for every n. For each such set of configurations $\mathbf{b}$, there exists a function $h_{\mathbf{b}}$, in a particular class of functions, such that $H(\mathbf{b}, k_s, k_b) = \mathscr{H}(\mathbf{b}, h_{\mathbf{b}}, k_s, k_b)$, which leads to the second main theorem of this work.

Theorem B. *Let $\{u_\ell\}$ be a sequence of minimizers for J_{ε_ℓ}, for which $\mathbf{a} = (a_1, \ldots, a_d)$ is a limiting configuration of distinct defects as $\varepsilon_\ell \to 0$ as described in Theorem A* *and $h \in H^1(\Omega)$ is as in Theorem A. Then it holds that $H(\mathbf{a},k_s,k_b) = \mathscr{H}(\mathbf{a},h,k_s,k_b)$ and*

$$\lim_{\ell\to\infty}\left(J_{\varepsilon_\ell}(u_\ell) - \underline{k}\pi d\ln\left(\frac{1}{\varepsilon_\ell}\right)\right) = \underline{k}W(\mathbf{a}) + H(\mathbf{a},k_s,k_b) + d\gamma$$

where γ is a fixed constant associated with each of the defect core's energy. Moreover, the renormalized energy attains its minimum among $\mathbf{b} \in \Omega^d$ with distinct components at $\mathbf{b} = \mathbf{a}$.

The term $\underline{k}\pi d\ln(\frac{1}{\varepsilon})$ represents the energy around the vortices to leading order. The limit as ε tends to zero of the difference between this term and (1.1) gives the remaining energy over the domain Ω minus the vortices, with their location $\mathbf{a}$, minimizing (1.5). For the case $k_s = k_b$ the proofs for Theorems A and B follow from the results in [1] and [6] if Ω is simply connected, and their proofs can be extended if the domain is multiply connected. Theorem B allows us to characterize the limiting pattern, $u_*(x)$, near each a_m. This follows from the fact that h minimizes (1.7). For the case $k_s = k_b$ this implies that h is a harmonic function such that $v = g$ on $\partial\Omega$. Thus

$$u_*(\rho y + a_m) \to \beta_m y$$

as $\rho \to 0$ for each $y \in \partial B_1(0)$ where $\beta_m = e^{i(h(a_m)+\zeta(a_m)+\sum_{n\neq m}\theta_{a_n}(a_m))}$. For $k_s \neq k_b$ we find a much different structure. In this case the integral in (1.7) involving either the term curl v or div v must be finite, and as a result pins the values of h near each a_m so that

$$u_*(\rho y + a_m) \to \begin{cases} \pm y & \text{if } k_s < k_b \\ \pm iy & \text{if } k_b < k_s \end{cases}$$

in $L^2(\partial B_1(0);\mathbb{C})$ as $\rho \to 0$. Thus if $k_s < k_b$ the limiting texture u_* has a pure splay pattern near each defect and if $k_b < k_s$ then u_* asymptotically has a pure bend pattern near each a_m.

1.2. Applications

Smectic C materials are made of layers of liquid crystal molecules that pack so that their long axes form a fixed angle $0 < \theta_0 < \pi/2$ with the layer normal. The pattern is described using the layer structure and a director field $n(x)$ for the liquid crystal. The director is a unit vector field that lies parallel to the local average of the molecular long axes at x. One can then express $n(x) = \cos(\theta_0)\nu(x) + \sin(\theta_0)c(x)$ where $\nu(x)$ and $c(x)$ each are unit vector fields that are respectively parallel and perpendicular to the layer normal at x. These two fields are the fundamental unknowns that are used to characterize the material's configuration [7]. For the case of a thin film the layers are planer, given by the domain $\Omega \subset \mathbb{R}^2$ such that $\nu(x) = \langle 0,0,1\rangle$ and $c(x) = \langle c^1(x), c^2(x), 0\rangle$ with $x = (x_1,x_2)$. The film can be just several layers thick and the elastic energy for the liquid crystal pattern is given by the Oseen-Frank energy. Each layer in a smectic C (Sm C) liquid crystal can be represented as a two-dimensional liquid [8] and the integral is taken over the film,

$$\frac{1}{2}\int_\Omega \left(k_s(\operatorname{div} c)^2 + k_b(\operatorname{curl} c)^2\right) dx. \tag{1.8}$$

A Sm C* liquid crystal additionally forms a spontaneous polarization field that produces elastic and electro-static contributions to the energy. The polar field generates an elastic stress on the film whose effect is modeled by introducing boundary values for $c(x) = g(x)$ on $\partial\Omega$; see [4]. The field induces an electro-static contribution that appears in our energy by increasing the splay constant k_b above its bare elastic value [9] and this is a motivation for studying the case $k_s \neq k_b$.

If a smoke or dust particle lands on a free-standing film a defect forms in the film's texture. The particle induces a singularity in the spontaneous polarization field that in turn causes an island, several layers thicker than the film, to form around the defect. The island's shape eventually stabilizes and the island migrates within the film so as to reduce the total energy. Various experiments have been conducted and models put forward to investigate this phenomenon. See [4, 9, 10, 11, 12]. In these papers the notion of the c director is generalized to allow for defects. The experiments reported in [4, 9, 10, 11] indicate that a stable island is disk-like and that the c-director is tangential at the island's edge, so that the winding number of c on the edge of the disk is $+1$. In [4], Lee et al. model the island–defect ensemble by setting $\Omega = B_R(0) \setminus B_\varepsilon(0)$ with the defect represented by the ε void at the origin and investigate numerically the stability of rotationally invariant equilibria for (1.8). Their simulations for the case $k_s < k_b$ and ε sufficiently small, indicate that minimizers for (1.8) over $H^1(\Omega;\mathbb{S}^1)$ subject to tangential boundary conditions on $\partial B_R(0)$ form a simple spiral, turning from the tangential pattern at the edge of the disk to radial near the defect at the center.

In this paper we follow an order parameter approach as in [12] for the energy (1.1). The unknown field $u(x)$ is taken to be a generalization of the c director. In this case u need not have unit length and vanishes at a defect where the smectic order is allowed to melt. This description, in contrast to the one above, does not presuppose the nature or location of individual defects. We can apply Theorem A to obtain information on minimizers for the problem of an island, $\Omega = B_R(0)$ with the tangential boundary values $g(x) = \pm\frac{x^\perp}{R}$ (where $x^\perp = (-x_2, x_1) \equiv ix$). If $k_s < k_b$ and ε is taken sufficiently small it follows that a minimizer has one defect with degree $+1$ in $B_R(0)$, moreover the minimizer's pattern is near radial in a neighborhood of the defect. This is consistent with what was observed in the experiments and simulations from [4].

In [12], Silvestre et al. investigate a different aspect of the problem. They consider a free standing SmC^* film occupying a simply connected region D containing d disjoint circular islands $\{\overline{B}_{R_j}(x_j);\ 1 \le j \le d\}$ and they investigate the texture in the background film $\Omega \equiv D \setminus \bigcup_{j=1}^{d} \overline{B}_{R_j}(x_j)$. In this case $\partial\Omega$ has $d+1$ components, ∂D and $\partial B_{R_j}(x_j)$ for $1 \le j \le d$. On ∂D they take $g =$ const. and $g = \pm\frac{(x-x_j)^\perp}{R_j}$ on $\partial B_{R_j}(x_j)$ for $1 \le j \le d$. It follows that $\deg(g, \partial\Omega) = -d$. The simulations and experiments in [12] exhibit d topological defects in Ω, each with degree -1, as companions to the d chiral islands. Our results do not directly apply to this setting. For the case of the classic Ginzburg-Landau energy (2.7), structure proved for the case $\deg(g, \partial\Omega) > 0$ directly translates to the same information for the case $\deg(g, \partial\Omega) < 0$. This is not obvious for the energy (1.1). Nevertheless the present analysis should be useful for investigating the case of boundary data with negative degree.

Our paper is organized as follows. In Section 2, we prove Theorem A, developing a number of qualitative features for the minimizers of J_ε. We express the integral in the form of (1.2) – (1.3). Written in this way one sees that minimizers form a family of low energy states for the Ginzburg-Landau functional (2.7), that is a family $\{u_\varepsilon\} \subset H^1_g(\Omega)$ satisfying (2.9) for a fixed constant K. It is proved in [13, 14, 15] that such a family has a number of structure and compactness properties, in particular it is shown that for a sequence $\varepsilon_k \to 0$, there exists a subsequence $\{u_{\varepsilon_{k(\ell)}}\}$ and a function u_* such that $u_{\varepsilon_{k(\ell)}} \to u_*$. The analysis in [15], due to Fanghua Lin, contains the detailed description of u_* that is needed here and we expand on this work in Proposition 2.3. We then use the fact that the u_ε are minimizers to refine the notion of convergence away from the defects of u_*. Our work here builds on the investigation of minimizers for the Ginzburg-Landau energy (2.7) carried out by Brezis-Bethuel-Hélein in [1]. Their work however relies on a priori estimates for $\sup|u_\varepsilon|$, which they obtain by applying a maximum principle that is not available in the case at hand. Here we proceed as in [16], obtaining bounds on the bulk term of the energy, giving us a priori L^4 estimates for the sequence of minimizers. A uniform bound on $\sup|u_\varepsilon|$ follows from this and elliptic estimates. In Section 3 we analyze a class of polar functions for a particular set of points $\mathbf{b} = (b_1, \ldots, b_d)$ and show that the spherical average of these functions around b_n tend to either 0 or $\frac{\pi}{2}$, mod π, depending on the relative values of k_s and k_b. In Section 4, we construct the renormalized energy in (1.5). We show that for each $\mathbf{a}$, there exists a polar function $h_{\mathbf{a}}$ that minimizes the renormalized energy for $\mathbf{a}$. In Section 5, we prove Theorem B.

2. Qualitative Properties of Minimizers

We begin by developing a number of structural properties for minimizers of (1.1) for each ε. As stated before, the minimizers in $H^1_g(\Omega;\mathbb{R}^2)$ of (1.1) are also minimizers in $H^1_g(\Omega;\mathbb{R}^2)$ of (1.3). In translating the variational problem into minimizing the energy (1.3) in $H^1_g(\Omega;\mathbb{R}^2)$, we have a strictly convex integrand (in the gradient), giving us the existence of a minimizer to our problem [5]. Then, taking the first variation of (1.3), we get that the minimizer u_ε satisfies

$$\begin{aligned}
&\int_\Omega \Big((k_s u^1_{x_1}) v^1_{x_1} + (k_s u^1_{x_2} + (k_b - k_s)(u^1_{x_2} - u^2_{x_1})) v^1_{x_2} + (k_s u^2_{x_1} + (k_b - k_s)(u^2_{x_1} - u^1_{x_2})) v^2_{x_1} \\
&\qquad + (k_s u^2_{x_2}) v^2_{x_2} + \frac{1}{\varepsilon^2}(u(1-|u|^2)) \cdot v \Big)\, dx = 0 \ \text{ if } \underline{k} = k_s \\
&\int_\Omega \Big((k_b u^1_{x_1} + (k_s - k_b)(u^1_{x_1} + u^2_{x_2})) v^1_{x_1} + (k_b u^1_{x_2}) v^1_{x_2} + (k_b u^2_{x_1}) v^2_{x_1} \\
&\qquad + (k_b u^2_{x_2} + (k_s - k_b)(u^1_{x_1} + u^2_{x_2})) v^2_{x_2} + \frac{1}{\varepsilon^2}(u(1-|u|^2)) \cdot v \Big)\, dx = 0 \ \text{ if } \underline{k} = k_b
\end{aligned} \tag{2.1}$$

for $v \in H^1_0(\Omega;\mathbb{R}^2)$. From regularity theory [17, 18], we have $u_\varepsilon \in C^\infty(\Omega) \bigcap C^{2,\alpha}(\overline{\Omega})$ for each $0 < \alpha < 1$ and $\varepsilon > 0$, since $\partial\Omega$ is C^3, $g \in C^3$, and the coefficients of the elliptic elastic energy term are constant. Thus we conclude that $|u_\varepsilon|_{2,\alpha;\overline{\Omega}} \le C(\varepsilon)$. Using integration by parts, we have that the Euler-Lagrange equation the minimizer satisfies is

$$\begin{aligned} -k_s\Delta u^1 - (k_b-k_s)(u^1_{x_2x_2} - u^2_{x_1x_2}) &= \frac{1}{\varepsilon^2}u^1(1-|u|^2) \\ -k_s\Delta u^2 - (k_b-k_s)(u^2_{x_1x_1} - u^1_{x_2x_1}) &= \frac{1}{\varepsilon^2}u^2(1-|u|^2) \end{aligned} \tag{2.2}$$

if $\underline{k} = k_s$, and

$$\begin{aligned} -k_b\Delta u^1 - (k_s-k_b)(u^1_{x_1x_1} + u^2_{x_2x_1}) &= \frac{1}{\varepsilon^2}u^1(1-|u|^2) \\ -k_b\Delta u^2 - (k_s-k_b)(u^2_{x_2x_2} + u^1_{x_1x_2}) &= \frac{1}{\varepsilon^2}u^2(1-|u|^2) \end{aligned} \tag{2.3}$$

if $\underline{k} = k_b$ in Ω, with $u = g$ on $\partial\Omega$. For each $\varepsilon > 0$, there is an upper and a lower bound that can be obtained for the integral (1.3).

Proposition 2.1.

$$\bar{J}_\varepsilon(u_\varepsilon) \le \underline{k}\pi d\ln(\frac{1}{\varepsilon}) + C_2(\Omega, g, d, k_s, k_b) \tag{2.4}$$

for a minimizer u_ε and

$$\bar{J}_\varepsilon(u) \ge \underline{k}\pi d\ln(\frac{1}{\varepsilon}) - C_1(\Omega, g, d, k_s, k_b) \tag{2.5}$$

for any function $u \in H^1_g$ where C_1 and C_2 are positive constants.

Proof. For the upper estimate, this proof is similar to the proof of Lemma 2.1 in [6]. Consider the case when $\Omega = B_R(0) = B_R$ and $g(x) = \beta\frac{x}{|x|}$, $\beta = \pm 1$ if $\underline{k} = k_s$ or $\beta = \pm i$ if $\underline{k} = k_b$. We drop the ε for notational purposes. Denote the values

$$\bar{I}_\beta(\varepsilon, R) = \inf_{u \in H^1_g}\left\{\int_{B_R(0)} \bar{j}_\varepsilon(u, \nabla u)\,dx\right\}. \tag{2.6}$$

Let $\bar{I}_\beta(t) = \bar{I}_\beta(t,1)$. By using a change of variables, $\bar{I}_\beta(\varepsilon, R) = \bar{I}_\beta(1, \frac{R}{\varepsilon}) = \bar{I}_\beta(\frac{\varepsilon}{R})$. Then, using the same method as in the proof of Theorem 3.1 in [1], noting that $\mathrm{div}(\frac{ix}{|x|}) = \mathrm{curl}(\frac{x}{|x|}) = 0$, we have $\bar{I}_\beta(t_1) \le \underline{k}\pi\ln(\frac{t_2}{t_1}) + \bar{I}_\beta(t_2)$ for all $t_1 \le t_2$.

Fix d points, $a_1, a_2, \dots, a_d, a_n \ne a_m$ for $n \ne m$ in Ω and $R > 0$ such that

$$\overline{B_R(a_n)} \subset \Omega \text{ for each } n \text{ and } \overline{B_R(a_n)} \cap \overline{B_R(a_m)} = \emptyset \text{ for every } n \ne m.$$

Let $\Omega_R = \Omega \setminus \bigcup_{n=1}^d B_R(a_n)$ and consider $\bar{g}(x) : \partial\Omega_R \to \mathbb{S}^1$ such that

$$\bar{g}(x) = \begin{cases} g(x), & \text{if} \quad x \in \partial\Omega \\ e^{i\theta}, & \text{if} \quad x = a_j + Re^{i\theta} \in \partial B_R(a_j) \text{ for some } j \text{ and } \underline{k} = k_s \\ ie^{i\theta}, & \text{if} \quad x = a_j + Re^{i\theta} \in \partial B_R(a_j) \text{ for some } j \text{ and } \underline{k} = k_b. \end{cases}$$

Note that $\deg(\bar{g}, \partial\Omega_R) = 0$, hence there exists a smooth function $v : \bar{\Omega}_R \to \mathbb{S}^1$ such that $v|_{\partial\Omega_R} = \bar{g}$. Then by the above claim for $0 < \varepsilon < R$,

$$\begin{aligned} \bar{J}_\varepsilon(u) &\leq \int_{\Omega_R} \bar{j}_\varepsilon(v, \nabla v)\, dx + \sum_{i=1}^{d} \bar{I}_\beta(\varepsilon, R) \\ &\leq \underline{k}\pi d \ln(\frac{1}{\varepsilon}) + C_2. \end{aligned}$$

Define

$$F_\varepsilon(u) = F_\varepsilon(u;\Omega) = \frac{1}{2}\int_\Omega \left(\underline{k}|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right) dx. \tag{2.7}$$

Now, let $u \in H^1_g$ be any function. Note that $\bar{J}_\varepsilon(u) \geq F_\varepsilon(u)$. The lower bound (2.5) holds for a minimizer $v_\varepsilon \in H^1_g$ for $F_\varepsilon(\cdot)$ and is proved in [1] for the case that Ω is star shaped. The general case follows from [6] and [1]. (See [13] and[14] for alternative proofs.) It follows that the lower bound holds for any $u \in H^1_g$. □

The following corollary is a direct result of the previous proposition, utilizing a method described in [14].

Corollary 2.1. *If u is a minimizer for $\bar{J}_\varepsilon(u)$, then there exists $C_4 = C_4(g, \Omega, k_s, k_b, d)$ such that*

$$\begin{aligned} &\int_\Omega \left((k_b - k_s)(\text{curl } u)^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2)\right) dx \leq C_4 \;\text{ if } \underline{k} = k_s, \\ &\int_\Omega \left((k_s - k_b)(\text{div } u)^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2)\right) dx \leq C_4 \;\text{ if } \underline{k} = k_b. \end{aligned} \tag{2.8}$$

Proof. For clarity, we provide a short proof. From Proposition 2.1, we have that for any minimizer $u_\varepsilon \in H^1_g$, $\bar{J}_\varepsilon(u_\varepsilon) \leq \underline{k}\pi d \ln(\varepsilon^{-1}) + C_2$. Let $F_{2\varepsilon}(u)$ be as defined in the proof of Proposition 2.1. Then $F_{2\varepsilon}(u_\varepsilon) \geq \underline{k}\pi d \ln(\varepsilon^{-1}) - C_3$. Thus (2.8) follows from considering the expression $\bar{J}_\varepsilon(u_\varepsilon) - F_{2\varepsilon}(u_\varepsilon)$. □

Using the estimate from the corollary and the strong ellipticity of the system in (2.2) and (2.3), we obtain several bounds for minimizers over the entire domain for small ε. Let $x \in \overline{\Omega}$ and set

$$\tilde{\Omega} = \{y : \varepsilon y + x \in \Omega\} \quad \text{and } \tilde{u}(y) = u_\varepsilon(\varepsilon y + x) \text{ for } y \in \tilde{\Omega}.$$

Then from (2.8) we have that $||\tilde{u}||_{4;\tilde{\Omega}\bigcap B_1(0)} \leq C$ where C is independent of $x \in \overline{\Omega}$ and $0 < \varepsilon \leq 1$. If we express (2.2) and (2.3) as $\mathscr{L}_{\underline{k}} u = \varepsilon^{-2}\mathbf{f}(u)$, where $\mathbf{f}(u) = u(1 - |u|^2)$, then $\mathscr{L}_{\underline{k}}$ is a second order strongly elliptic operator with constant coefficients. Moreover we have that $\mathscr{L}_{\underline{k}}\tilde{u} = \mathbf{f}(\tilde{u})$ on $\tilde{\Omega}\bigcap B_1(0)$. Based on the L^4 a priori estimate, the ellipticity of the operator $\mathscr{L}_{\underline{k}}$, and the smoothness of both the boundary data and $\partial\Omega$, the proof of the next proposition follows just as the proof for Lemma 3.1 in [16].

Proposition 2.2. *There exists a constant $C_5 = C_5(g, \Omega, k_s, k_b, d)$ so that if u_ε is a minimizer to $\bar{J}_\varepsilon(u)$, then*

$$|u_\varepsilon|, \varepsilon|\nabla u_\varepsilon| \leq C_5 \quad \text{in } \Omega, \text{ for } 0 < \varepsilon < 1.$$

From Proposition 2.1 and the definition of $\bar{j}_\varepsilon(u,\nabla u)$, we get that

$$F_\varepsilon(u_\varepsilon) \le \underline{k}\pi d \ln(\frac{1}{\varepsilon}) + K \tag{2.9}$$

for every $0 < \varepsilon < 1$ and minimizer u_ε to $\bar{J}_\varepsilon(\cdot)$. With this estimate we are able to apply a Structure and Compactness theorem to the sequence of minimizers for $\bar{J}_\varepsilon(u)$.

Proposition 2.3. *There exists constants $\delta > 0$ and C depending on $K, \underline{k}, g$, and Ω so that for any sequence of functions $u_\varepsilon \in H^1_g(\Omega;\mathbb{R}^2)$ with $\varepsilon \downarrow 0$ satisfying* (2.9) *there exists a subsequence $\{u_{\varepsilon_\ell}\}$, points $\{a_1,\dots,a_d\} \subset \Omega$, and a function $h(x) \in H^1(\Omega)$ so that*

$$\min\{dist(a_m,\partial\Omega), |a_m - a_n|, m \ne n, 1 \le m,n \le d\} \ge \delta, \quad \|h\|_{H^1} \le C,$$

and

$$u_{\varepsilon_\ell} \to \prod_{m=1}^{d} \frac{x-a_m}{|x-a_m|} e^{i(h(x)+\zeta(x))}$$

where the convergence of $\{u_{\varepsilon_\ell}\}$ is weak in $H^1_{loc}(\overline{\Omega} \setminus \{a_1,\dots,a_d\};\mathbb{C})$ and strong in $L^2(\Omega;\mathbb{C})$.

Proposition 2.3 is due to Fanghua Lin [3, 15, 19, 20] if Ω is simply connected. In this case the limit takes the form $\prod_{m=1}^{d} \frac{x-a_m}{|x-a_m|} e^{ih(x)}$. We modify his arguments below to prove it for general Ω. We first need a lemma.

Lemma 2.1. *There is a constant K', depending only on $K, \underline{k}, g$, and Ω so that if u satisfies* (2.9) *then $w^*(x)u(x) = e^{-i\zeta(x)}u(x)$ satisfies*

$$F_\varepsilon(w^*u) \le \underline{k}\pi d \ln\left(\frac{1}{\varepsilon}\right) + K'.$$

Proof. For $a \in \mathbb{C}$ we denote a^* as the complex conjugate of a. Writing $wu = e^{i\zeta}u$. We have

$$F_\varepsilon(wu) = F_\varepsilon(u) + \underline{k}\int_\Omega \mathscr{I}m\{u^*\nabla\zeta \cdot \nabla u\}\,dx + \frac{k}{2}\int_\Omega |\nabla\zeta|^2|u|^2\,dx.$$

Since $deg(wu;\partial\Omega) = d$ we have the lower bound

$$F_\varepsilon(wu) \ge \underline{k}\pi d \ln\left(\frac{1}{\varepsilon}\right) - C$$

where C depends on $wu|_{\partial\Omega} = wg$. Second, since ζ is smooth and fixed in $\overline{\Omega}$ we have that

$$\int_\Omega |\nabla\zeta|^2|u|^2\,dx \le M\int_\Omega |u|^2\,dx \le M_1\left(\varepsilon^2 \ln\left(\frac{1}{\varepsilon}\right) + 1\right).$$

With these two estimates, together with our hypothesis on $F_\varepsilon(u)$ we get

$$-\int_\Omega \mathscr{I}m\{u^*\nabla\zeta \cdot \nabla u\}\,dx \le C$$

where C depends only on $K, \underline{k}, g$, and Ω. Finally we can expand and estimate

$$\begin{aligned} F_\varepsilon(w^* u) &= F_\varepsilon(u) - \underline{k}\int_\Omega \mathscr{I}m\{u^*\nabla\zeta\cdot\nabla u\}\,dx + \frac{k}{2}\int_\Omega |\nabla\zeta|^2|u|^2\,dx \\ &\le \underline{k}\pi d \ln\left(\frac{1}{\varepsilon}\right) + K'. \end{aligned}$$

□

Proof of Proposition 2.3. Let $u_\varepsilon \in H^1_g(\Omega;\mathbb{R}^2)$ with $\varepsilon \downarrow 0$ satisfying (2.9). Set $z_\varepsilon = w^* u_\varepsilon$. Then the winding number for $z_\varepsilon|_{\Gamma_j}$ is 0 for $1 \le j \le k$ and we can extend z_ε onto Λ_j as a function in $H^1(\Lambda_j;\mathbb{S}^1)$ that is independent of ε for each j. We also have that the winding number for $z_\varepsilon|_{\Gamma_0}$ is d. Thus setting $z_\varepsilon|_{\Gamma_0} = \tilde{g}$, we have that the sequence $\{z_\varepsilon\} \subset H^1_{\tilde{g}}(D;\mathbb{C})$ and that

$$F_\varepsilon(z_\varepsilon;D) \le \underline{k}\pi d \ln\left(\frac{1}{\varepsilon}\right) + K''.$$

where K'' is independent of ε. We can apply the Proposition for the simply connected domain D; see [15]. We find a subsequence $\{z_{\varepsilon_\ell}\}$, points $\{a_1,\dots,a_d\} \subset D$, and $h \in H^1(D)$ so that

$$z_{\varepsilon_\ell} \to z_*(x) = \prod_{m=1}^{d} \frac{x-a_m}{|x-a_m|} e^{ih(x)} \qquad \text{in } D.$$

Thus since $u_{\varepsilon_\ell} = w z_{\varepsilon_\ell}$ we have that

$$u_{\varepsilon_\ell} \to \prod_{m=1}^{d} \frac{x-a_m}{|x-a_m|} e^{i(h(x)+\zeta(x))} \qquad \text{in } \Omega.$$

We know that the $\{a_1,\dots,a_d\}$ are uniformly bounded away from each other and Γ_0. It remains to show that they are bounded away from $\bigcup_{j=1}^{k} \overline{\Lambda}_j$. If this is not so, then we can find a case with $a_\ell \in \overline{\Lambda}_j$ for some ℓ and j. We choose $r > 0$ sufficiently small so that $\overline{B}_r(a_\ell) \bigcap \{a_n : n \neq \ell\} = \emptyset$. By construction $z_{\varepsilon_\ell}(x)$ for $x \in \Lambda_j$ are independent of ε_ℓ and in $H^1(\Lambda_j)$. Thus $z_* \in H^1(\Lambda_j)$. On the other hand we have

$$z_*(x) = \frac{x-a_\ell}{|x-a_\ell|}\tilde{z}(x) \text{ such that } |\tilde{z}(x)| = 1 \text{ for } x \in B_r(a_\ell). \text{ Moreover } \tilde{z} \in H^1(B_r(a_\ell)).$$

Thus $z_* \notin H^1(\Lambda_j)$ and this is a contradiction. □

For $\rho > 0$ set $\Omega_\rho = \Omega \setminus \bigcup_{m=1}^{d} \overline{B}_\rho(a_m)$.

Proposition 2.4. *Let $\{u_{\varepsilon_\ell}\}$ be a sequence of minimizers converging to $u_*(x) = \prod_{m=1}^{d} \frac{x-a_m}{|x-a_m|} e^{i(h(x)+\zeta(x))}$ in $L^2(\Omega)$. Then for each $\rho > 0$ the convergence is in $H^1(\Omega_\rho)$ and $\varepsilon_\ell^{-2}\int_{\Omega_\rho}(1-|u_{\varepsilon_\ell}|^2)^2\,dx \to 0$ as $\ell \to \infty$. Moreover, u_* is a local minimizer for the limiting energy in $H^1(\Omega_\rho,\mathbb{S}^1)$.*

Proof. The proof is similar to the proof of Lemma 3.9 in [16]. For notational purposes we write $\{u_\ell\} = \{u_{\varepsilon_\ell}\}$, where ε_ℓ is a subsequence of ε and $\varepsilon \to 0$. By Proposition 2.3, $u_\ell \rightharpoonup u_*$ in $H^1(\Omega_\rho)$ for every $\rho > 0$ and u_ℓ is a local minimizer for $\int_{\Omega_\rho} \bar{j}_\varepsilon(u, \nabla u)\, dx$. As in the proof of Lemma 3.9 of [16], choose $\bar{x} \in \bar{\Omega} \setminus \{a_1, \ldots, a_d\}$, assuming first that $\bar{x} \notin \partial\Omega$ and let $\bar{d} = \bar{d}(\bar{x})$ be such that $\bar{B}_{2\bar{d}} = \bar{B}_{2\bar{d}}(\bar{x}) \subset \Omega \setminus \{a_1, \ldots, a_d\}$. Set $\omega(x) = h(x) + \zeta(x) + \sum_{n=1}^{d} \theta_{a_n}(x)$ where $\frac{x-a_m}{|x-a_m|} = e^{i\theta_{a_m}(x)}$. Then with out loss of generality ω is single valued in $B_{2\bar{d}}$. Furthermore $\omega \in H^1(B_{2\bar{d}})$ by Proposition 2.3 and $u_*(x) = e^{i\omega(x)}$ on $\bar{B}_{2\bar{d}}$. From Corollary 2.1, Proposition 2.3, and as in the proof of Lemma 3.9 of [16], for a subsequence $\{u_{\ell'}\}$ that we do not relabel, there exists a radius d such that $\bar{d} \le d \le 2\bar{d}$ and for which

$$\frac{1}{2}\int_{\partial B_d} \left(\underline{k}|\partial_\tau u_\ell|^2 + \frac{1}{2\varepsilon_\ell^2}(1-|u_\ell|^2)^2 \right) d\sigma \le C_1. \tag{2.10}$$

It follows that $\{u_\ell\}$ converges to u_* uniformly on ∂B_d and weakly in $H^1(\partial B_d)$. Note that $\deg u_*\big|_{\partial B_d} = 0$. Therefore $\deg u_\ell\big|_{\partial B_d} = 0$ for sufficiently large ℓ. Thus $u_\ell(x)$ can be expressed as $u_\ell(x) = |u_\ell(x)|e^{i\omega_\ell(x)}$ such that $|u_\ell(x)| \ne 0$ for every $x \in \partial B_d$ where ω_ℓ converges to ω uniformly on ∂B_d and weakly in $H^1(\partial B_d)$ as well.

Define, for each ℓ, the function $\Phi_\ell(r,\theta) := \phi_\ell(r)(\omega_\ell(\theta) - \omega(d,\theta))$, where $d \ge r = |x - \bar{x}|$,

$$\phi_\ell(r) = \begin{cases} 1 & \text{if } r = d \\ 0 & \text{if } r < r_\ell < d \end{cases}$$

ϕ_ℓ is a smooth cut-off function and $|\nabla \phi_\ell| \le \frac{1}{d - r_\ell}$, where r_ℓ is a sequence of radii such that $r_\ell \to d$ so that

$$\int_{\partial B_d} \frac{|\omega_\ell - \omega|^2}{(r_\ell - d)}\, d\sigma \to 0.$$

By construction and the fact that $\omega_\ell \rightharpoonup \omega$ in $H^1(\partial B_d)$, we get that $\Phi_\ell \to 0$ in $H^1(B_d)$ and $\Phi_\ell \to 0$ uniformly in B_d. Let $\bar{u}$ minimize $\int_{B_d} \bar{j}(u, \nabla u)\, dx$ in the set $\{u \in H^1(B_d; \mathbb{S}^1) \,:\, u = u_* \text{ on } \partial B_d\}$. Consider the function $e^{i\Phi_\ell}\bar{u}$. Then we have $e^{i\Phi_\ell}\bar{u} \to \bar{u}$ in $H^1(B_d)$, and uniformly in $\overline{B}_d$. Now, we construct comparison functions

$$\hat{u}_\ell = |\hat{u}_\ell| e^{i\Phi_\ell}\bar{u} \quad \text{on } B_d$$

where

$$|\hat{u}_\ell| = 1 \qquad \text{on } B_{d-\varepsilon_\ell},$$
$$|\hat{u}_\ell| = |u_\ell| \qquad \text{on } \partial B_d,$$

and for each θ, define $|\hat{u}_\ell|(|x-\bar{x}|, \theta)$ to be linear for $d - \varepsilon_\ell \le |x - \bar{x}| \le d$. This gives that $\hat{u}_\ell = u_\ell$ on ∂B_d. By construction, we get that $\hat{u}_\ell \to \bar{u}$ uniformly in $\overline{B_d}$, $\hat{u}_\ell \to \bar{u}$ in $H^1(B_d)$, and $\varepsilon_\ell^{-2} \int_{B_d} (1 - |\hat{u}_\ell|^2)^2\, dx \to 0$ as $\ell \to \infty$. These limits imply that

$$\lim_{\ell \to 0} \int_{B_d} \bar{j}_{\varepsilon_\ell}(\hat{u}_\ell, \nabla \hat{u}_\ell)\, dx = \int_{B_d} \bar{j}(\bar{u}, \nabla \bar{u})\, dx := \bar{J}(\bar{u})$$

where

$$\bar{j}(u, \nabla u) = \begin{cases} k_s|\nabla u|^2 + (k_b - k_s)(\text{curl } u)^2 & \text{if } \underline{k} = k_s \\ k_b|\nabla u|^2 + (k_s - k_b)(\text{div } u)^2 & \text{if } \underline{k} = k_b. \end{cases}$$

This notion for $\bar{j}(u,\nabla u)$ will be used throughout the rest of the work. Then by the lower semicontinuity of the integral $\int_\Omega \bar{j}(u,\nabla u)\,dx$,

$$\begin{aligned}\int_{B_d} \bar{j}(u_*,\nabla u_*)\,dx &\le \liminf_{\ell\to\infty}\int_{B_d} \bar{j}_{\varepsilon_\ell}(u_\ell,\nabla u_\ell)\,dx \le \limsup_{\ell\to\infty}\int_{B_d} \bar{j}_{\varepsilon_\ell}(u_\ell,\nabla u_\ell)\,dx \\ &\le \lim_{\ell\to\infty}\int_{B_d} \bar{j}_{\varepsilon_\ell}(\hat{u}_\ell,\nabla \hat{u}_\ell)\,dx = \int_{B_d} \bar{j}(\bar{u},\nabla\bar{u})\,dx \\ &\le \int_{B_d} \bar{j}(u_*,\nabla u_*)\,dx.\end{aligned}$$

This implies that

$$\lim_{\ell\to\infty}\int_{B_d} \bar{j}_{\varepsilon_\ell}(u_\ell,\nabla u_\ell)\,dx = \int_{B_d} \bar{j}(u_*,\nabla u_*)\,dx.$$

Since $u_\ell \rightharpoonup u_*$ in $H^1(B_d;\mathbb{C})$, then by [21], $u_\ell \to u_*$ in $H^1(B_d;\mathbb{C})$. A further consequence of the convergence of the integrals and the strong convergence in $H^1(B_d;\mathbb{C})$ is

$$\lim_{\ell\to\infty}\varepsilon_\ell^{-2}\int_{B_d}(1-|u_\ell|^2)^2\,dx = 0.$$

We have also showed that u_* is a minimizer to $\bar{J}$ in $H^1_{u_*}(B_d;\mathbb{S}^1)$. We have proved our assertions for a subsequence of the original sequence on $B_d \supset B_{\overline{d}}$, for some radius d. It follows that we have established the assertions for the full sequence on B_d.

Finally, suppose $\bar{x}\in\partial\Omega$. Then let $U_{\bar{x}}$ be a neighborhood of $\bar{x}$ such that there exists a smooth diffeomorphism $\psi(x)$ defined on $B_{2\overline{d}}$ such that $\psi(\bar{x}) = 0$ and

$$\psi : U_{\bar{x}} \to B^+_{2\overline{d}} = \{(x,y) : x^2+y^2 < (2\overline{d})^2, y>0\}.$$

Then we can carry out the same argument in $B^+_{2\overline{d}}$, with $\psi(\bar{x})$, push back into $U_{\bar{x}}$, and then argue as in the previous case. □

This next proposition shows that the norms of the minimizers converge uniformly to 1 outside of any positive radius distance away from the vortices. This proof is similar to the proof of Lemma 3.10 in [16].

Proposition 2.5. *Let $\{u_\ell\}$ be a sequence of minimizers with $\varepsilon_\ell \to 0$, converging to $u_*(x) = \prod_{m=1}^d \frac{x-a_m}{|x-a_m|}e^{(ih(x)+\zeta(x))}$ in $L^2(\Omega)$. Then for each $\rho>0, |u_\ell|\to 1$ uniformly in $\overline{\Omega}_\rho$.*

Proof. Fix $\rho>0$ and assume that there exists a $\delta>0$, a subsequence $\{u_\ell\}$ (that we do not relabel), and a sequence of points $\{x_\ell\}\subset\overline{\Omega}_\rho$ so that $|1-|u_\ell(x_\ell)||\ge\delta$. From Proposition 2.2 there is a $c(\delta)>0$ so that $|1-|u_\ell(x)||\ge\frac{\delta}{2}$ for $x\in B_{c\varepsilon_\ell}(x_\ell)\bigcap\Omega$. It follows that

$$\varepsilon_\ell^{-2}\int_{\Omega_\rho}(1-|u_{\varepsilon_\ell}|^2)^2\,dx \ge \varepsilon_\ell^{-2}\int_{B_{c\varepsilon_\ell}(x_\ell)\bigcap\Omega_\rho}(1-|u_{\varepsilon_\ell}|^2)^2\,dx \ge \overline{C} > 0$$

where $\overline{C}$ is independent of ℓ. We have seen from Proposition 2.4 however that the left side tends to 0 as $\ell\to\infty$ and this leads to a contradiction.

□

The next two propositions prove higher regularity on the sequence of minimizers on compact subsets of the domain away from the vortices. For the sequence of minimizers $\{u_\ell\}$ that converges in $H^1_{loc}(\overline{\Omega}\setminus\{a_1,\dots,a_d\})$, we have that $|u_\ell|$ converges uniformly to 1 on every $K \subset\subset \overline{\Omega}\setminus\{a_1,\dots,a_d\}$. The bulk term of the energy has a non-degenerate minimum when $|u| = 1$. The elastic term of the energy is strongly elliptic, as well as quadratic in the gradient term. Due to these facts, the proofs of the following propositions follow from the proofs of Lemma 3.11 and Lemma 3.12 from [16] respectively.

Proposition 2.6. *Let $\{u_{\varepsilon_\ell}\}$ be a sequence of minimizers for* (1.1) *in $H^1_g(\Omega)$ converging in $H^1_{loc}(\overline{\Omega}\setminus\{a_1,a_2,\dots,a_d\})$ as $\varepsilon_\ell \to 0$. Then, for $K \subset\subset \overline{\Omega}\setminus\{a_1,a_2,\dots,a_d\}$, there exists constants ℓ_0 and C such that if $\ell \geq \ell_0$, then*

$$\|D^2 u_{\varepsilon_\ell}\|_{2;K} \leq C.$$

Proposition 2.7. *Let u_{ε_ℓ} be the sequence of minimizers as in the previous lemma. For each $n > 2$ and set $K \subset\subset \Omega\setminus\{a_1,a_2,\dots,a_d\}$, there are constants C and ℓ_0 such that*

$$\|u_{\varepsilon_\ell}\|_{n,2;K} \leq C \text{ for } \ell \geq \ell_0.$$

A consequence of Propositions 2.6 and 2.7 is that for each compact subset K of $\Omega\setminus\{a_1,\dots,a_d\}$ and for each $n \geq 2$, we have the entire converging subsequence of minimizers $\{u_{\varepsilon_\ell}\} \in H^n(K)$ and bounded. This implies that $u_* \in H^n(K)$. Now, by Sobolev's Theorem and Arzelà-Ascoli Theorem, we have the following corollary

Corollary 2.2. *Let $\{u_{\varepsilon_\ell}\}$ be a sequence of minimizers converging to u_*. Then for any $0 < \alpha < 1$, and each integer m*

$$u_{\varepsilon_\ell} \to u_* \quad \text{in} \quad C^\alpha_{loc}(\overline{\Omega}\setminus\{a_1,\dots,a_d\}) \text{ and } C^m_{loc}(\Omega\setminus\{a_1,\dots,a_d\})$$

as $\ell \to \infty$.

Proof of Theorem A. Let $\{u_\varepsilon\}$ be a sequence of minimizers to (1.1) such that $\varepsilon \downarrow 0$. Then we know it is also a minimizer of (1.3) for each ε. By applying Proposition 2.3, it follows that there exists a subsequence $\{u_{\varepsilon_\ell}\}$, a function h, and points $\{a_1,\dots,a_d\}$ such that

$$u_{\varepsilon_\ell}(x) \to \prod_{n=1}^{d} \frac{x-a_n}{|x-a_n|} e^{i(h(x)+\zeta(x))} = u_*(x) \quad \text{in } H^1_{loc}(\overline{\Omega}\setminus\{a_1,a_2,\dots,a_d\}).$$

For each $\rho > 0$ it follows from Corollary 2.2 that we have $u_{\varepsilon_\ell} \to u_*$ in $C^\alpha(\overline{\Omega_\rho})$ for every $0 < \alpha < 1$, and $u_{\varepsilon_\ell} \to u_*$ in $C^m(\overline{\Omega}_\rho)$ for every integer $m \geq 0$, where $\Omega_\rho = \Omega\setminus\cup_{n=1}^d \overline{B}_\rho(a_n)$. □

Remark 1. *To be definite we point out that Theorem A applies to the case $k_s = k_b > 0$, and that in this case our arguments are extensions of those from* [1], [6], *and* [15] *that allow us to treat the case of a multiply connected domain. Moreover, in this case the nature of $h(x)$ is distinct. Indeed from Proposition 2.4 we have that h is harmonic in $\Omega\setminus\{a_1\dots,a_d\}$ and $h \in H^1(\Omega)$. As such h is harmonic in Ω. (This observation originates in* [1]*).*

In the next section we show that in contrast with the case $k_s = k_b$, the values of h are pinned at $\{a_1\dots,a_d\}$ if $k_s \neq k_b$.

3. Class of Functions for Each Configuration of Points

Assume that $k_s \neq k_b$. Define the set

$$T = \{\mathbf{b} = (b_1, \ldots, b_d) \in \Omega^d \,:\, b_n \neq b_m \text{ for } n \neq m\}.$$

Fix a configuration $\mathbf{b} \in T$. We can choose $f \in C^3(\partial\Omega)$ so that

$$g(x) = e^{i(f(x)+\sum_{n=1}^{d}\theta_{b_n}(x)+\zeta(x))} \quad for\; x \in \partial\Omega$$

where θ_{b_n} is such that $\frac{x-b_n}{|x-b_n|} = e^{i\theta_{b_n}(x)}$ for $x \neq b_n$. Note that $f|_{\Gamma_\ell}$ is uniquely determined, mod 2π, for each component Γ_ℓ of $\partial\Omega$. Let $\phi \in H^1(\Omega)$ and set

$$v(x) = v(\mathbf{b},\phi)(x) = e^{i(\phi(x)+\sum_{n=1}^{d}\theta_{b_n}(x)+\zeta(x))}.$$

We define the set

$$A(\mathbf{b}) = \begin{cases} \{\phi \in H^1(\Omega) \,:\, v(\mathbf{b},\phi) = g \text{ on } \partial\Omega \text{ and } \int_\Omega (\text{curl } v)^2\,dx < \infty\} \text{ if } k_s < k_b \\ \{\phi \in H^1(\Omega) \,:\, v(\mathbf{b},\phi) = g \text{ on } \partial\Omega \text{ and } \int_\Omega (\text{div } v)^2\,dx < \infty\} \text{ if } k_b < k_s. \end{cases}$$

It follows from Lemma 1.1 of [22] that $\phi = f + 2\pi t_\ell$ on each component Γ_ℓ of $\partial\Omega$, for some $t_\ell \in \mathbb{Z}$, for each $\phi \in A(\mathbf{b})$. We prove in Proposition 4.1 that $A(\mathbf{b})$ is nonempty. Let $\rho > 0$ be such that $B_\rho(b_n) \subset \Omega$ and such that $\{\overline{B}_\rho(b_n)\}$ are pairwise disjoint. For $h \in A(\mathbf{b})$ define the function $h_n(x) = h(x) + \sum_{m\neq n}\theta_{b_m}(x) + \zeta(x)$ for $x \in B_\rho(b_n)$. Set $v = v(\mathbf{b},h)$. From the definition of $A(\mathbf{b})$, we have

$$\int_{B_\rho(b_n)} (\text{curl } v)^2\,dx \leq C \text{ if } k_s < k_b,$$
$$\int_{B_\rho(b_n)} (\text{div } v)^2\,dx \leq C \text{ if } k_b < k_s,$$

for each $1 \leq n \leq d$ such that $C = C(v) < \infty$. On $B_\rho(b_n)$ we have that $v = \frac{x-b_n}{|x-b_n|}e^{ih_n} = \cos(h_n)\frac{x-b_n}{|x-b_n|} + \sin(h_n)i\frac{x-b_n}{|x-b_n|}$. Since $h_n \in H^1(B_\rho(b_n))$, curl $\frac{x-b_n}{|x-b_n|} = \text{div } i\frac{x-b_n}{|x-b_n|} = 0$, and div $\frac{x-b_n}{|x-b_n|} = \text{curl } i\frac{x-b_n}{|x-b_n|} = \frac{1}{|x-b_n|}$, we obtain using Young's Inequality that

$$\begin{aligned} &\int_{B_\rho(b_n)} \frac{\sin^2(h_n)}{|x-b_n|^2}\,dx \leq C \text{ if } k_s < k_b, \\ &\int_{B_\rho(b_n)} \frac{\cos^2(h_n)}{|x-b_n|^2}\,dx \leq C \text{ if } k_b < k_s. \end{aligned} \tag{3.1}$$

From (3.1), we obtain the following proposition.

Proposition 3.1. *Let* $\mathbf{b} \in T$ *and* $h \in A(\mathbf{b})$. *Then we have*

$$\frac{1}{|\partial B_\rho(b_n)|}\int_{\partial B_\rho(b_n)} h\,d\sigma \to \alpha_n - \sum_{m\neq n}\theta_{b_m}(b_n) - \zeta(b_n) \quad \text{as} \quad \rho \to 0,$$

for each $1 \leq n \leq d$ *where*

$$\alpha_n = \begin{cases} c_n\pi & \textit{for some } c_n \in \mathbb{Z} \textit{ if } k_s < k_b \\ \dfrac{(2c_n+1)\pi}{2} & \textit{for some } c_n \in \mathbb{Z} \textit{ if } k_s < k_b. \end{cases}$$

Proof. Let $h_n(x) = h(x) + \sum_{m\neq n} \theta_{b_m}(x) + \zeta(x)$ and

$$\bar{h}_n(\rho) = \frac{1}{|\partial B_\rho(b_n)|}\int_{\partial B_\rho(b_n)} h_n\, d\sigma.$$

Define the functions $\omega^n(x) = \sin(h_n(x))$, $\bar{\omega}^n(\rho) = \sin(\bar{h}_n(\rho))$ if $k_s < k_b$, and $\omega^n(x) = \cos(h_n(x))$, $\bar{\omega}^n(\rho) = \cos(\bar{h}_n(\rho))$ if $k_b < k_s$. Set $\omega^n_\rho(y) = \omega^n(\rho y + b_n)$ for $0 < \rho < \rho_0$ where ρ_0 is such that $B_{2\rho_0}(b_n) \subset\subset \Omega \setminus \bigcup_{m\neq n} B_{2\rho_0}(b_m)$. With $h_n \in H^1(B_{\rho_0}(b_n))$ and (3.1) we get

$$\lim_{\rho\to 0} \|\omega^n_\rho\|^2_{1,2;B_1} = \lim_{\rho\to 0}\int_{B_\rho(b_n)} \left(|\nabla\omega^n|^2 + \frac{|\omega^n|^2}{\rho^2}\right) dx = 0. \tag{3.2}$$

Note that $\bar{h}_n(\rho) \in C((0,\rho_0])$. If the proposition is false there exists a constant $\delta_0 > 0$ and a sequence of radii $\{\rho_k\}$ such that $\rho_k \to 0$ and

$$\begin{aligned} |\bar{h}_n(\rho_k) - t\pi| \geq \delta_0 > 0 \text{ if } k_s < k_b,\\ |\bar{h}_n(\rho_k) - \frac{(2t+1)\pi}{2}| \geq \delta_0 > 0 \text{ if } k_b < k_s, \end{aligned} \tag{3.3}$$

for every $t \in \mathbb{Z}$. Since $\int_{B_2} |\nabla_y \Big(h_n(\rho y + b_n)\Big)|^2\, dy \to 0$ as $\rho \to 0$, we have

$$\lim_{\rho\to 0}\{\sup_{1/2\leq s\leq 1}\int_{\partial B_1} |h_n(s\rho y + b_n) - \bar{h}_n(\rho)|^2\, d\sigma_y\} = 0. \tag{3.4}$$

Then from the Lipschitz continuity of the sine and cosine functions and (3.4) we have

$$\lim_{\rho_k\to 0}\{\sup_{1/2\leq s\leq 2}\int_{\partial B_1} |\omega^n_{s\rho_k}(y) - \bar{\omega}^n(\rho_k)|^2\, d\sigma_y\} = 0. \tag{3.5}$$

From (3.3) we get

$$|(\bar{\omega}^n(\rho_k))| \geq C_3 > 0 \tag{3.6}$$

for each k. Then, from (3.5) and (3.6) we get for every $s \in [1/2, 1]$ and every k sufficiently large that

$$\int_{\partial B_s} |\omega^n_{\rho_k}|^2\, d\sigma \geq C_4 > 0.$$

This implies

$$\begin{aligned} \int_{B_{\rho_k}(b_n)} \frac{|\omega^n|^2}{\rho_k^2}\, dx &\geq \int_{B_{\rho_k}(b_n)\setminus B_{\rho_k/2}(b_n)} \frac{|\omega^n|^2}{\rho_k^2}\, dx\\ &= \int_{\frac{1}{2}}^1 \int_{\partial B_s} |\omega^n_{\rho_k}|^2\, d\sigma ds \geq \frac{C_4}{4} \end{aligned}$$

which is a contradiction. □

Using the notation from Proposition 3.1 we have the following corollary.

Corollary 3.1. *Let* $h_n(x) = h(x) + \sum_{m\neq n} \theta_{b_m}(x) + \zeta(x)$ *for* $h \in A(\mathbf{b})$. *Then for* $1 \leq n \leq d$

$$\lim_{\rho\to 0}\int_{B_\rho(b_n)\setminus B_{\rho/2}(b_n)} \left(|\nabla h_n|^2 + \frac{(h_n - \alpha_n)^2}{\rho^2}\right) dx = 0$$

and

$$h_n(\rho y+b_n)\to\alpha_n \quad \text{in } L^2(\mathbb{S}^1) \text{ as } \rho\to 0.$$

From the definition of α_n then, for each $\phi\in A(\mathbf{b})$, if $k_s<k_b$ and $v(x)=v(\mathbf{b},\phi)(x)$ we have $v(\rho y+b_n)\to\pm y$ as $\rho\to 0$ in $L^2(\partial B_1(0),\mathbb{S}^1)$, giving rise to a pure splay pattern near each defect. If $k_b<k_s$ it follows that $v(\rho y+b_n)\to\pm iy$ in $L^2(\partial B_1(0),\mathbb{S}^1)$, giving rise to a pure bend pattern near each defect.

4. Construction and Properties of the Renormalized Energy

Denote the integrals $J_\varepsilon(u;A)=\int_A j_\varepsilon(u,\nabla u)\,dx$ and $\bar{J}_\varepsilon(u;A)=\int_A \bar{j}_\varepsilon(u,\nabla u)\,dx$, with $A\subset\Omega$, noting that $J_\varepsilon(u;\Omega)=J_\varepsilon(u)$. Given any configuration $\mathbf{b}\in T$, take a function $\phi\in A(\mathbf{b})$ and define the function v as in the previous section. Then, for a given $\rho>0$, denoting $\Omega_\rho=\Omega\setminus\cup_{n=1}^d\overline{B}_\rho(b_n)$, we have

$$\bar{J}_\varepsilon(v;\Omega_\rho)=\bar{J}(v;\Omega_\rho)=\int_{\Omega_\rho}\bar{j}(v,\nabla v)\,dx$$

where

$$v=e^{i(\phi+\zeta+\sum_{n=1}^d\theta_{b_n})} \text{ in } \Omega_\rho.$$

Then, we have

$$\frac{k}{2}\int_{\Omega_\rho}|\nabla v|^2\,dx=\frac{k}{2}\int_{\Omega_\rho}\Big(|\nabla\phi|^2+2\nabla\phi\cdot(\nabla\zeta+\sum_{n=1}^d\nabla\theta_{b_n})+|\nabla\zeta+\sum_{n=1}^d\nabla\theta_{b_n}|^2\Big)\,dx.$$

Recall that $\theta_{b_n}(x)$ is the harmonic conjugate of $\ln(|x-b_n|)$, $\zeta(x)$ is the harmonic conjugate of $-\sum_{\ell=1}^k d_\ell\ln(|x-y_\ell|)$ in Ω_ρ, and $\{y_1,\dots,y_k\}\subset(\overline{\Omega})^c$. Define a function $G_\mathbf{b}(x)=\sum_{n=1}^d\ln(|x-b_n|)-\sum_{\ell=1}^k d_\ell\ln(|x-y_\ell|)$. Hence, using integration by parts, we obtain

$$\begin{aligned}\frac{k}{2}\int_{\Omega_\rho}|\nabla v|^2\,dx=&\frac{k}{2}\int_{\Omega_\rho}|\nabla\phi|^2\,dx+\frac{k}{2}\int_{\partial\Omega}\Big((\partial_\nu G_\mathbf{b})G_\mathbf{b}-2(\partial_\tau G_\mathbf{b})\phi\Big)\,d\sigma\\&-\sum_{m\neq n}\underline{k}\pi\ln(|b_n-b_m|)+\sum_{n=1}^d\sum_{\ell=1}^k\underline{k}\pi d_\ell\ln(|b_n-y_\ell|)+\underline{k}\pi d\ln\left(\frac{1}{\rho}\right)\\&+o_\rho(1).\end{aligned}$$

We also have on the boundary, $v(x)=e^{i(\phi(x)+\sum\theta_{b_n}(x)+\zeta(x))}=g(x)$. Using integration by parts again, we have

$$\begin{aligned}\frac{k}{2}\int_{\Omega_\rho}|\nabla v|^2\,dx=&\frac{k}{2}\int_{\Omega_\rho}|\nabla\phi|^2\,dx+\frac{k}{2}\int_{\partial\Omega}\Big(2G_\mathbf{b}(g\times\partial_\tau g)-(\partial_\nu G_\mathbf{b})G_\mathbf{b}\Big)\,d\sigma\\&-\sum_{m\neq n}\underline{k}\pi\ln(|b_n-b_m|)+\sum_{n=1}^d\sum_{\ell=1}^k\underline{k}\pi d_\ell\ln(|b_n-y_\ell|)+\underline{k}\pi d\ln\left(\frac{1}{\rho}\right)\\&+o_\rho(1).\end{aligned}$$

This implies that

$$\begin{aligned} J(v;\Omega_\rho) &= \bar{J}(v;\Omega_\rho) + \underline{k}\pi d \\ &= \underline{k}\pi d \ln\left(\frac{1}{\rho}\right) + \underline{k}W(\mathbf{b}) + \mathscr{H}(\mathbf{b},\phi,k_s,k_b) + o_\rho(1) \end{aligned} \tag{4.1}$$

where $W(\mathbf{b})$ and $\mathscr{H}(\mathbf{b},h,k_s,k_b)$ are defined as in (1.6) and (1.7) respectively. Notice that $\mathscr{H}(\mathbf{b},\phi,k_s,k_b) \geq \mathscr{H}(\mathbf{b},\phi,\underline{k},\underline{k})$ and that $\underline{k}W(\mathbf{b}) + H(\mathbf{b},\underline{k},\underline{k})$ is simply $\underline{k}$ times the renormalized energy for the Ginzburg-Landau energy studied in [1] for the case when Ω is simply connected. It is proved there that the latter tends to infinity as either $\mathbf{b} \to \partial\Omega^d$ or $|b_n - b_m| \to 0$. These properties can be seen to hold here by examining the first and third terms on the right side of (1.6) respectively. Thus they hold for our renormalized energy as well. To minimize the energy then, the vortices must be distinct and stay within the domain Ω. Now for each configuration, we will show that there is a function that will minimize the renormalized energy.

Proposition 4.1. *Assume that $k_s \neq k_b$. Let $\mathbf{b} \in T$ be a configuration in Ω^d, Then, there exists a function $h_{\mathbf{b}}(x) \in A_{\mathbf{b}}$ such that*

$$\begin{aligned} \underline{k}W(\mathbf{b}) + H(\mathbf{b},k_s,k_b) &= \min_{\phi\in A(\mathbf{b})} \left(\underline{k}W(\mathbf{b}) + \mathscr{H}(\mathbf{b},\phi,k_s,k_b)\right) \\ &= \underline{k}W(\mathbf{b}) + \mathscr{H}(\mathbf{b},h_{\mathbf{b}},k_s,k_b). \end{aligned}$$

Proof. We first point out that $A(\mathbf{b}) \neq \emptyset$. Indeed let $\phi \in H^1(\Omega)$ such that $v(\mathbf{b},\phi) = g$ on $\partial\Omega$. Let $\{\overline{B}_\rho(b_n); n = 1,\ldots,d\}$ be a nonintersecting collection of closed disks that are contained in Ω. One can always modify ϕ so that $\phi(x) = \alpha_n - \sum_{m\neq n}\theta_{b_m}(x) - \zeta(x)$ for $x \in \overline{B}_\rho(b_n)$, for each n, for some α_n as defined in Proposition 3.1. The resulting function $\phi \in A(\mathbf{b})$.

Since $W(\mathbf{b})$ is independent of the particular $\phi \in A(\mathbf{b})$ we only need to minimize $\mathscr{H}(\mathbf{b},\cdot,k_s,k_b)$. Let $\{h_n\} \subset A(\mathbf{b})$ be a minimizing sequence for $\mathscr{H}(\mathbf{b},\cdot,k_s,k_b)$. By definition of the integral, we can subtract an integer multiple of 2π from each h_n, so that without a loss of generality $h_n|_{\Gamma_0} = f$ for each n. Then Poincaré's inequality can be applied so that we have $\|h_n\|_{2;\Omega} \leq C(\|\nabla h_n\|_{2;\Omega} + 1)$. Since $\{h_n\}$ is a minimizing sequence for $\mathscr{H}(\mathbf{b},\cdot,k_s,k_b)$ it follows that $\|h_n\|_{1,2;\Omega} \leq C_0$ for some constant C_0. Thus there exists a function $h_0 \in H^1(\Omega)$ satisfying $v(\mathbf{b},h_0) = g$ on $\partial\Omega$ and such that $h_{n_k} \rightharpoonup h_0$ in $H^1(\Omega)$ and $h_{n_k} \to h_0$ almost everywhere in Ω. This gives

$$\liminf_{k\to\infty} \int_\Omega |\nabla h_{n_k}|^2\,dx \geq \int_\Omega |\nabla h_0|^2\,dx. \tag{4.2}$$

This also gives $e^{ih_{n_k}(x)} \rightharpoonup e^{ih_0(x)}$ in $H^1(\Omega;\mathbb{C})$. Let $\prod_{n=1}^d \frac{x-b_n}{|x-b_n|} e^{i(h_{n_k}(x)+\zeta(x))} = w_{n_k}(x)$ and $\prod_{n=1}^d \frac{x-b_n}{|x-b_n|} e^{i(h_0(x)+\zeta(x))} = w_0(x)$. Then we have $w_{n_k} \to w_0$ in $L^2(\Omega,\mathbb{C})$. Let $z_k =$ curl w_{n_k}(div w_{n_k}) if $k_s < k_b(k_b < k_s)$. Then, we have $\|z_k\|_{2;\Omega} \leq C(\mathbf{b})$. Hence, there exists a subsequence, relabeled as z_k, such that $z_k \rightharpoonup z_0$ in $L^2(\Omega)$. This implies that $z_0 =$ curl w_0(div w_0) and we have

$$\begin{aligned} \liminf_{k\to\infty} \int_\Omega (\text{curl } w_{n_k})^2\,dx &\geq \int_\Omega (\text{curl } w_0)^2\,dx \quad \text{if } k_s < k_b, \\ \liminf_{k\to\infty} \int_\Omega (\text{div } w_{n_k})^2\,dx &\geq \int_\Omega (\text{div } w_0)^2\,dx \quad \text{if } k_b < k_s. \end{aligned} \tag{4.3}$$

Then combining (4.2) and (4.3), we get

$$\min_{h\in H_v^1}(W(\mathbf{b})+\mathscr{H}(\mathbf{b},h,k_s,k_b)) = \liminf_{k\to\infty}(W(\mathbf{b})+\mathscr{H}(\mathbf{b},h_{n_k},k_s,k_b)) \\ \geq W(\mathbf{b})+\mathscr{H}(\mathbf{b},h_0,k_s,k_b)$$

giving us the result of the proof, with $h_{\mathbf{b}} := h_0$. □

By the previous proposition, we can utilize the notation as in (1.5) so that there is no dependency on the choice of the function h. Even though it will not be used in this work, one can show that $\underline{k}W(\mathbf{b})+H(\mathbf{b},k_s,k_b)$ is continuous on T. This is done in [23].

Proposition 4.2. *Let h be the polar function appearing in the definition of u_* from Theorem A. Then $h\in A(\mathbf{a})$.*

Proof. Consider the configuration $\mathbf{a}=(a_1,\dots,a_d)$ and the function $u_* = \prod_{n=1}^d \frac{x-a_n}{|x-a_n|}e^{i((h(x)+\zeta(x))}$. Let $\{u_\ell\}$ be a sequence of minimizers to $J_{\varepsilon_\ell}(\cdot)$ for each ε_ℓ. Then from Corollary 2.1 we get $\int_\Omega(\text{curl } u_\ell)^2\,dx \leq \tilde{C}$ or $\int_\Omega(\text{div } u_\ell)^2\,dx \leq \tilde{C}$, where $\tilde{C}$ has no dependency on ε. From Proposition 2.3, $u_\ell \to u_*$ in $L^2(\Omega;\mathbb{C})$ and we have that the distributions $\{\text{curl } u_\ell\}$ ($\{\text{div } u_\ell\}$) are uniformly bounded in $L^2(\Omega)$. Hence, as in the proof of Proposition 4.1, we get

$$\int_\Omega(\text{curl } u_*)^2\,dx \leq \liminf_{\ell\to\infty}\int_\Omega(\text{curl } u_\ell)^2\,dx \leq C_0, \text{ if } k_s<k_b,$$
$$\int_\Omega(\text{div } u_*)^2\,dx \leq \liminf_{\ell\to\infty}\int_\Omega(\text{div } u_\ell)^2\,dx \leq C_0, \text{ if } k_b<k_s.$$

From the definition of each u_ℓ on the boundary, we have $u_*|_{\partial\Omega}=g$, giving us $v(\mathbf{b},h)=g$ on $\partial\Omega$. From Proposition 2.3, $h\in H^1(\Omega)$. Therefore, $h\in A(\mathbf{a})$ and must satisfy the result of Proposition 3.1. □

5. Energy Away From the Vortices

We must first analyze the following minimum problem before proving Theorem B. Let $\beta\in\mathbb{C}$, such that

$$\beta = \begin{cases} \pm 1 & \text{if } k_s<k_b \\ \pm i & \text{if } k_b<k_s \end{cases}$$

and recall from the proof of Proposition 2.1 the expression

$$\bar{I}_\beta(\frac{\varepsilon}{R}) = \inf_{u\in H_g^1}\left\{\int_{B_R(0)} \bar{j}_\varepsilon(u,\nabla u)\,dx\right\}$$

where

$$g(x)=\beta\frac{x}{|x|} \text{ for } |x|=R.$$

Proposition 5.1. *The function $\bar{I}_\beta(\tau)+\underline{k}\pi\ln(\tau)$ is a nondecreasing function of τ for $\tau>0$ such that $\gamma := \lim_{\tau\to 0}\{\bar{I}_\beta(\tau)+\underline{k}\pi\ln(\tau)\} > -\infty$.*

Proof. The argument is similar to that in [1]. It is shown within the proof of Proposition 2.1 that the expression $\bar{I}_\beta(\tau)+\underline{k}\pi\ln(\tau)$ is monotone nondecreasing and bounded below. The existence of the finite one-sided limit at $\tau=0$ follows from these two properties. □

Proof of Theorem B. Assume that $k_s \neq k_b$. We argue in a similar manner as in [16] and Chapter 8 of [1]. Those works however, used the fact that the polar function $h_{\mathbf{b}}(x)$ is smooth (since it is harmonic for the case $k_s = k_b$). Here we appeal to Corollary 3.1 to control $h_{\mathbf{b}}$. Let $\mathbf{b}=(b_1,\dots,b_n)$ be a configuration in Ω^d, where $b_n \neq b_m$ for $m \neq n$, and $\Omega_\rho = \Omega \setminus \cup_{n=1}^d \overline{B}_\rho(b_n)$. For this configuration, set

$$w_{\mathbf{b}}(x) = \prod_{n=1}^{d} \frac{x-b_n}{|x-b_n|} e^{i(h_{\mathbf{b}}(x)+\zeta(x))}$$

where $h_{\mathbf{b}} \in A(\mathbf{b})$ satisfies Proposition 4.1. Then, using (4.1) we find that,

$$\int_{\Omega_\rho} \bar{j}(w_{\mathbf{b}},\nabla w_{\mathbf{b}})\,dx = \underline{k}\pi d \ln\left(\frac{1}{\rho}\right) + \underline{k}W(\mathbf{b}) + H(\mathbf{b},k_s,k_b) - \underline{k}\pi d + o_\rho(1) \tag{5.1}$$

as $\rho \to 0$. We next construct comparison functions using the configuration point $\mathbf{b}$. For $0<\varepsilon \ll \rho \ll 1$, define

$$\tilde{u}_\ell(x) = \begin{cases} w_{\mathbf{b}}(x) & \text{for } x \in \Omega_\rho \\ e^{iq_n(x)}\dfrac{x-b_n}{|x-b_n|} & \text{for } x \in B_\rho(b_n)\setminus B_{\rho/2}(b_n) \\ z_n(x-b_n) & \text{for } x \in B_{\rho/2}(b_n) \end{cases}$$

where z_n minimizes $\bar{J}_{\varepsilon_\ell}(\cdot;B_{\rho/2}(0))$, with

$$z_n(x-b_n)|_{\partial B_{\rho/2}(b_n)} = e^{i\alpha_n}\frac{x-b_n}{|x-b_n|}.$$

Here α_n is determined from $h_{\mathbf{b}}$ by way of Corollary 3.1 such that

$$\alpha_n = \begin{cases} c_n\pi & \text{for some } c_n \in \mathbb{Z} \text{ if } k_s<k_b \\ \dfrac{(2c_n+1)\pi}{2} & \text{for some } c_n \in \mathbb{Z} \text{ if } k_b<k_s. \end{cases}$$

Then by Proposition 5.1, we have

$$\int_{B_{\rho/2}(0)} \bar{j}_\varepsilon(z_n,\nabla z_n)\,dx = \underline{k}\pi \ln\left(\frac{\rho}{2\varepsilon}\right) + \gamma + o_\varepsilon(1) \tag{5.2}$$

as $\varepsilon \to 0$ for each $\rho>0$. Define $\tilde{h}_n(x) = h_{\mathbf{b}}(x) + \sum_{m\neq n}\theta_{b_m}(x) + \zeta(x)$ and q_n such that

$$q_n(x)|_{\partial B_\rho} = \tilde{h}_n(x), \quad q_n(x)|_{\partial B_{\rho/2}} = \alpha_n$$

such that q_n is linear in radial directions centered at b_n. From Corollary 3.1 there exist $\rho_\ell \downarrow 0$ so that

$$\lim_{\ell\to\infty}\int_{\partial B_{\rho_\ell}(b_n)} \left(\rho_\ell|\nabla\tilde{h}_n|^2 + \frac{(\tilde{h}_n-\alpha_n)^2}{\rho_\ell}\right)d\sigma = 0. \tag{5.3}$$

With this property we have that

$$\tilde{u}_\ell(x) = e^{iq_n(x)}\frac{x-b_n}{|x-b_n|} = (e^{i(q_n(x)-\alpha_n)})(e^{i\alpha_n}\frac{x-b_n}{|x-b_n|}) \quad \text{for } x \in B_{\rho_\ell}(b_n)\setminus B_{\rho_\ell/2}(b_n)$$

satisfies

$$\int_{B_{\rho_\ell}\setminus B_{\rho_\ell/2}} \bar{j}_\varepsilon(\tilde{u}_\ell,\nabla\tilde{u}_\ell)\,dx = \underline{k}\pi\ln(2)+o_\rho(1) \tag{5.4}$$

uniformly in ε. Then from (5.1), (5.2), and (5.4) we have for minimizers u_ε that

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &= \bar{J}_\varepsilon(u_\varepsilon)+\underline{k}\pi d \le \bar{J}_\varepsilon(\tilde{u}_\ell)+\underline{k}\pi d \\ &= \underline{k}\pi d\ln\left(\frac{1}{\varepsilon}\right)+\underline{k}W(\mathbf{b})+H(\mathbf{b},k_s,k_b)+d\gamma+o_\rho(1)+o_\varepsilon(1). \end{aligned}$$

Thus

$$\limsup_{\varepsilon\to 0}\left(J_\varepsilon(u_\varepsilon)-\underline{k}\pi d\ln\left(\frac{1}{\varepsilon}\right)\right) \le \underline{k}W(\mathbf{b})+H(\mathbf{b},k_s,k_b)+d\gamma. \tag{5.5}$$

We next obtain an estimate from below. Let $\mathbf{a}$ be a limiting configuration as in Theorem A. Using Propositions 2.4 and 4.2 we can find a sequence of radii $\{\rho_\ell\}$ and a subsequence of minimizers $\{u_{\varepsilon_{k,\ell}}\}$ (that we label $\{u_k\}$) so that (5.3) holds at $\mathbf{a}$, and in addition

$$u_k \to u_* \text{ in } H^1(\partial B_{\rho_\ell}(a_n)) \text{ and } \frac{1-|u_k|^2}{\varepsilon_k} \to 0 \text{ in } L^2(\partial B_{\rho_\ell}(a_n)) \text{ as } k\to\infty$$

for each ℓ and n. It follows that we can construct, in a similar fashion as before, functions $\tilde{u}_{nk\ell} \in H^1(B_{2\rho_\ell}(a_n)\setminus B_{\rho_\ell}(a_n);\mathbb{C})$ so that

$$\tilde{u}_{nk\ell}|_{\partial B_{\rho_\ell}(a_n)} = u_k \text{ and } \tilde{u}_{nk\ell}|_{\partial B_{2\rho_\ell}(a_n)} = e^{i\alpha_n}\frac{x-a_n}{|x-a_n|},$$

satisfying

$$\int_{B_{2\rho_\ell}\setminus B_{\rho_\ell}} \bar{j}_{\varepsilon_k}(\tilde{u}_{nk\ell},\nabla\tilde{u}_{nk\ell})\,dx = \underline{k}\pi\ln(2)+o_\rho(1)+o_\varepsilon(1).$$

From this and Proposition 5.1 we see that

$$\begin{aligned} \underline{k}\pi\ln\left(\frac{\rho_\ell}{\varepsilon_k}\right)+\gamma &\le \bar{J}_{\varepsilon_k}(u_k;B_{\rho_\ell})+\bar{J}_{\varepsilon_k}(\tilde{u}_{nk\ell};B_{2\rho_\ell}\setminus B_{\rho_\ell})-\underline{k}\pi\ln(2)+o_\rho(1)+o_\varepsilon(1) \\ &= \bar{J}_{\varepsilon_k}(u_k;B_{\rho_\ell})+o_\rho(1)+o_\varepsilon(1). \end{aligned}$$

We use Proposition 2.4 and (4.1) to determine the asymptotic nature for $\bar{J}_{\varepsilon_k}(u_k;\Omega_{\rho_\ell})$. These two estimates give

$$\begin{aligned} J_{\varepsilon_k}(u_k) &\ge \underline{k}\pi d\ln(\frac{1}{\varepsilon_k})+\underline{k}W(\mathbf{a})+\mathscr{H}(\mathbf{a},h,k_s,k_b)+d\gamma+o_\rho(1)+o_\varepsilon(1) \\ &\ge \underline{k}\pi d\ln(\frac{1}{\varepsilon_k})+\underline{k}W(\mathbf{a})+H(\mathbf{a},k_s,k_b)+d\gamma+o_\rho(1)+o_\varepsilon(1). \end{aligned}$$

Choosing subsequences $\{\rho_\ell\},\{\varepsilon_\ell\}$ such that $\varepsilon_\ell \ll \rho_\ell \to 0$ as $\ell\to\infty$ allows us to compare this to (5.5). Since $\mathbf{b}$ was arbitrary, we have that

$$\underline{k}W(\mathbf{a})+H(\mathbf{a},k_s,k_b) \le \underline{k}W(\mathbf{a})+\mathscr{H}(\mathbf{a},h,k_s,k_b) \le \underline{k}W(\mathbf{b})+H(\mathbf{b},k_s,k_b)$$

for any configuration $\mathbf{b} \in T$. Hence the configuration $\mathbf{a}$ from Proposition 2.3 minimizes the renormalized energy, with $\mathbf{a} \in T$. If we set $\mathbf{a} = \mathbf{b}$, we obtain that the function h from Proposition 2.3 minimizes the renormalized energy for the configuration $\mathbf{a}$. Finally we see that

$$\lim_{\ell \to \infty} \left(J_{\varepsilon_\ell}(u_{\varepsilon_\ell}) - \underline{k}\pi d \ln\left(\frac{1}{\varepsilon_\ell}\right)\right) = \underline{k}W(\mathbf{a}) + H(\mathbf{a}, k_s, k_b) + d\gamma.$$

□

Remark 2. *To be complete, we point out that Theorem B holds in the case* $k = k_s = k_b$ *as well. In this case the renormalized energy is as in* (1.5) *and* (1.6)*, such that* $H(\mathbf{b}, k, k) = \frac{1}{2}\int_\Omega k|\nabla h_\mathbf{b}|^2\, dx$, *where* $v(\mathbf{b}, h_\mathbf{b}) = g$ *on* $\partial\Omega$ *and* $h_\mathbf{b}$ *is a harmonic function that minimizes this energy subject to this boundary condition. If* Ω *is multiply connected the proof proceeds just as in* [1]*, once one knows that a configuration* $\mathbf{a}$ *obtained from a sequence of minimizers* $\{u_\ell\}$ *is in* T*. This follows from Proposition 2.3.*

Acknowledgements

This research was supported by ED grant 84.200A, NSF Grant REC-0535794 and NSF Grant DMS-1109495

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