Some Dynamical Properties of Ginzburg–Landau Vortices

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Dedicated to L. Nirenberg and P. Lax on the occasions of their 70th birthdays, with respect and admiration

1. Introduction

We consider the vortex motion for the Ginzburg–Landau heat flow

$$u_t = \Delta u + \frac{1}{\varepsilon^2}(1 - |u|^2)u \quad \text{in } \Omega \times \mathbb{R}_+ , \tag{1.1}$$

$$u(x,t) = g(x) \quad \text{for } x \in \partial\Omega,\ t > 0 , \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega . \tag{1.3}$$

Here Ω is a two-dimensional, smooth, bounded domain, ε is a positive parameter, $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}^2$, $g : \partial\Omega \to \mathbb{R}^2$ is smooth, and $|g|(x) = 1, x \in \partial\Omega$. Naturally we also assume the compatibility condition that $u_0(x) = g(x)$ on $\partial\Omega$.

The system (1.1)–(1.3) can be viewed as a simplified evolutionary Ginzburg–Landau equation in the theory of superconductivity ([4], [5], [11], [18]). The same system also appears in a canonical way when one expands a large class of second-order dissipative systems about bifurcation points ([3], [15], [19]). It serves, therefore, as one of the fundamental models in the study of the dynamics of nonequilibrium patterns ([21], [22]).

The aim of this article is to understand the global (in time) dynamics of vortices, or zeros, of solutions u of (1.1)–(1.3). Our study has some interesting implications for the problem of "pinning the Ginzburg–Landau vortices"; see, for example, [6] and [16]. Its importance to the theory of superconductivity and applications are addressed in many earlier works ([6], [7], [10], [14], [18]).

To understand the behavior of solutions u of (1.1)–(1.3) as $t \to +\infty$, one has to look at steady state solutions u_ε, that is, the critical points of the energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(|u|^2 - 1\right)^2 \right] dx . \tag{1.4}$$

A complete characterization of asymptotic behavior (as $\varepsilon \to 0^+$) of vortices of u_ε is given in the recent book [2].

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THEOREM 1.1. *Let Ω, g be as above, and let $\{u_{\varepsilon_n}\}$ be a sequence of steady state solutions of* (1.1)–(1.2) *(with $\varepsilon = \varepsilon_n$). Then there is a subsequence $\{u_{\varepsilon_n'}\}$ such that $u_{\varepsilon_n'}(x) \to u^*(x)$ in $C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega}/\{a_1, \dots, a_k\})$ with $k \leqq k(\Omega, g)$ where $u^*(x)$ is given by*

$$u^*(x) = \prod_{j=1}^{k} \left(\frac{x - a_j}{|x - a_j|} \right)^{d_j} e^{ih(x)}, \quad x \in \Omega, \tag{1.5}$$

$\sum_{j=1}^k d_j = d \equiv \deg(g, \partial\Omega), u^*|_{\partial\Omega} = g$, and $\Delta h = 0$ in Ω. Here we assume $d \geqq 0$, and in the product we naturally identify a two-vector with a complex number.

If, in addition, the $\{u_{\varepsilon_n'}\}$ are minimizers of (1.4), *then, in the above formula for $u^*, k = d, d_j = 1$ for $j = 1, \dots, d$. Moreover, the point $a = (a_1, \dots, a_d) \in \Omega^d$ is a global minimum point of the renormalized energy $W_g(b)$ defined in $\overline{\Omega}^d$ where for $b = (b_1, \dots, b_d) \in \overline{\Omega}^d$,*

$$\begin{aligned} W_g(b) &= -\pi \sum_{j \neq i} \log |b_i - b_j| + \frac{1}{2} \int_{\partial\Omega} \Phi(g \wedge g_\tau) \\ &= -\pi \sum_{j=1}^{d} R(b_j), \end{aligned} \tag{1.6}$$

Φ is a solution of

$$\begin{aligned} \Delta\Phi &= 2\pi \sum_{j=1}^{d} \delta_{b_j} \quad \text{in } \Omega, \\ \frac{\partial\Phi}{\partial\nu} &= g \wedge g_\tau \quad \text{on } \partial\Omega, \end{aligned} \tag{1.7}$$

(g_τ is the tangential derivative of g along $\partial\Omega$), and $R(x)$ is given by

$$R(x) = \Phi(x) - \sum_{j=1}^{d} \log |x - b_j| . \tag{1.8}$$

Remark 1.2. The above theorem was shown in [2] under the additional assumption that Ω is star-shaped. This additional assumption was later removed in [16] and [25]. We point out that the proof of theorem A in [17] also leads to a proof of the latter fact.

The dynamics of the vortices in the limit $\varepsilon \to 0$ can be considered within the framework of a general program initiated by J. Neu [18] and later extended and improved by many others ([7], [20], [22]). They formally used the method of matched asymptotic expansions to derive equations of motion for the vortices. To leading order in ε (or $\frac{1}{\log \varepsilon}$), the equations are of the form:

$$m_i \frac{d}{dt} a_i(t) = -\nabla_{a_i} W_g(a), \quad i = 1, 2, \dots, d.$$

The constants m_i are called the *mobilities* of the vortices. One of the key facts that has been derived is that $m_i \sim |\log \varepsilon|$. In fact, it is derived in [18] and [7] that

$$\log \frac{1}{\varepsilon} \frac{d}{dt} a(t) = -\operatorname{grad} W_g(a) \tag{1.9}$$

where $a(t) = (a_1(t), \dots, a_d(t)), a_i(t)$, $i = 1, \dots, d$, are vortices.

It is a challenging problem to give a rigorous mathematical proof of (1.9). Since the right-hand side of (1.9) is generically bounded, it follows that the vortices move slowly; that is, it takes a period of time that is $O(\log \frac{1}{\varepsilon})$ for a vortex to move an appreciable distance. Under certain conditions on the initial data u_0 and the assumption that Ω is convex, the following result was proved in [22]:

THEOREM 1.3. *Let $u_\varepsilon(x, t)$ be a solution of* (1.1)–(1.3). *Assume that at each time $t > 0$ there exists exactly one zero of u_ε, denoted by $q_\varepsilon(t)$, with* $\deg(u_\varepsilon, \partial B_r =$ $(q_\varepsilon(t))$ *for all positive $r <$* $\operatorname{dist}(q_\varepsilon(t), \partial\Omega)$. *Let x_0 be any point in $\Omega|\{0\}$ and denote by T_ε the infinimum of the set $\{t : q_\varepsilon(t) = x_0\}$, assuming this set to be nonempty. Where $u_0(0) = 0$, then*

$$\liminf_{\varepsilon \to 0} \frac{T_\varepsilon}{|\log \varepsilon|} > 0.$$

The result of Theorem 1.3 shows that vortices can only move in no less than $\log \frac{1}{\varepsilon}$-scale time. It is, however, almost impossible to verify the assumption of this theorem in general. Indeed, in the same paper the authors exhibited examples in which the number of vortices increases in the course of the evolution. In Section 3 of this paper, we show, under a suitable condition on the initial data u_0, the solution $u_\varepsilon(x, t)$ will not create any additional vortices whenever $0 < t \leqq o(\log \frac{1}{\varepsilon})$.

The main results of this paper can be roughly described as follows. Let $u_\varepsilon(x, t)$ be the solution of (1.1)–(1.3), and suppose the initial data satisfy some natural conditions (see Assumptions 3.1 and 3.2 in Section 3); then, in any finite time T, we show the vortices of $u_\varepsilon(\cdot, t), 0 \leqq t \leqq T$, remain roughly the same as the initial data, and the phase function of $u_\varepsilon(\cdot, t)$ satisfies the standard heat equation (see Theorem 3.3) whenever ε is sufficiently small.

Next, we consider the evolution equations of the form

$$\frac{1}{\lambda_\varepsilon} \frac{\partial}{\partial t} u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon \left(1 - |u_\varepsilon|^2\right) \tag{1.10}$$

in $\Omega \times \mathbb{R}_+$ and u_ε satisfies (1.2) and (1.3). We show (see Theorem 3.7) that as $\varepsilon \to 0$, u_ε converges to

$$\prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{ih(x)}$$

in $L^2_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}_+)$ whenever

$$\lim_{\varepsilon \to 0} \frac{\lambda_\varepsilon}{\log \frac{1}{\varepsilon}} = 0 \quad \text{and} \quad \lambda_\varepsilon \to \infty \text{ as } \varepsilon \to 0\,.$$

Here $b_1, \dots, b_d$ are d distinct points in Ω. They are vortices for the initial data u_0 (see Assumptions 3.1 and 3.2) where $\Delta h(x) = 0$ in Ω and $h(x) = h_0(x)$ on $\partial\Omega$ so that

$$\prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{ih_0(x)} = g(x)\,.$$

The case that $\lambda_\varepsilon / \log \frac{1}{\varepsilon} \to \infty$ as $\varepsilon \to 0$ is studied in Section 4. Our main result there is Theorem 4.5, which states that, for any sequence of $\varepsilon_n \downarrow 0$, there is a subsequence of $\{u_{\varepsilon_n}(x,t)\}$ that converges in some generalized sense. Moreover, for a.e., $t > 0$, and any w-limits of $\{u_{\varepsilon_n}(\cdot, t)\}$ are functions of the form

$$\prod_{j=1}^{d} \frac{x-a_j}{|x-a_j|} e^{ih_a(x)}$$

with the property that $\Delta h_a = 0$ in $\Omega, h_a = h_0$ on $\partial\Omega$, and $a = (a_1, \dots, a_d)$ are critical points of $W_g(\cdot)$.

The proofs of these results are based on some basic facts concerning the function class $S_g(\lambda, K)$ (see Definition 2.3). Theorem 2.4, Theorem 3.3, and Lemma 4.1 (see also (4.5)) are each clearly of interest. They show that the function class $S_g(\lambda, K)$ possesses properties similar to those of functions U_ε that minimize the energy (1.4) and such that $U_\varepsilon = g$ on $\partial\Omega$ (cf. [2]).

As can be seen from the above discussion, the case $\lambda_\varepsilon \simeq \log \frac{1}{\varepsilon}$ is obviously most interesting for the motions of vortices. Though we are unable to show in this paper that the precise dynamical law (1.9) is valid, we can nevertheless show that (cf. Theorem 5.1) vortices move continuously in this time scale. (See the remark following this paper.) Moreover, vortices have to move at positive speeds whenever they are away from the critical point set of the renormalized energy. These statements show, in particular, that the mobilities of vortices have to be $\simeq \log \frac{1}{\varepsilon}$.

As our proofs are mostly based on a careful investigation into the class of functions $S_g(\lambda, K)$, the key method involved is naturally energy comparison. It is clear that, when $\varepsilon \to 0$, we will lose control on both continuity in time and continuity in space variables for the solutions $u_\varepsilon(x,t)$ of (1.1)–(1.3). Thus there is little hope that the standard elliptic or parabolic theory will have many implementations here.

2. Preliminaries

2.1. Maps with Prescribed Vortices

Let $b_1, b_2, \ldots, b_d$ be d distinct points in Ω where $d = \deg(g, \partial\Omega) > 0$. By lemma VIII.1 and its proof in [2], there is some ρ_0 depending only on $b = (b_1 \ldots, b_d)$ and Ω such that for every $0 < \rho \leqq \rho_0$ and every $\varepsilon > 0$, one may find some $w_\varepsilon(x) \in c^1(\Omega)$ with $w_\varepsilon = g$ on $\partial\Omega$ and

$$E_\varepsilon(w_\varepsilon) \leqq dI(\varepsilon, \rho) + W_g(b) + \pi d \log \frac{1}{\rho} + 0(\rho). \tag{2.1}$$

Moreover, $|\nabla w_\varepsilon(x)| \leqq c/\varepsilon$, $x \in \Omega$, and

$$\frac{1}{\varepsilon^2} \int_\Omega \left(|w_\varepsilon|^2 - 1\right)^2 dx \leqq c. \tag{2.2}$$

On the other hand, lemma VIII.2 of [2] asserts that

$$\begin{aligned} &\min\left\{E_\varepsilon(u) : u|_{\partial\Omega} = g\right\} \\ &\geqq dI(\varepsilon, \rho) + \pi d \log \frac{1}{\rho} + O(\rho^2) + \min\left\{W_g(a) : a \in \overline{\Omega}^d\right\} \end{aligned} \tag{2.3}$$

whenever $\varepsilon \leqq \varepsilon(\rho)$.

In inequalities (2.1) and (2.3), the quantity $I(\varepsilon, \rho)$ is defined by

$$I(\varepsilon, \rho) = \min\left\{\int_{B_\rho(0)} e_\varepsilon(u) dx : u(x) = \frac{x}{|x|} \text{ on } \partial B_\rho(0)\right\},$$

where $e_\varepsilon(u) = \frac{1}{2}\left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(|u|^2 - 1\right)^2\right]$. Maps that satisfy (2.1) and (2.2) will provide the particular class of initial data for our discussion below.

LEMMA 2.1. *Let u be a minimizer of the functional*

$$E_\varepsilon(u) = \frac{1}{2}\int_B \left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}\left(|u|^2 - 1\right)^2\right] dx$$

in the unit disc B with $u = g$ on ∂B. Suppose that

$$\int_{\partial B} \left[|g_\tau|^2 + \frac{1}{2\varepsilon^2}(|g|^2 - 1)^2\right] \leqq K,$$

for a constant K. Then, for all sufficiently small $\varepsilon > 0$ (depending only on K) we have

$$E_\varepsilon(u) \leqq C(K)$$

whenever $\deg(g, \partial B) = 0$, *and*

$$E_\varepsilon(u) \geqq \pi\,|d|\,\log\frac{1}{\varepsilon} - C(K)$$

if $\deg(g, \partial B) = d \neq 0$.

Proof: See lemma 1 of [17].

LEMMA 2.2. *Using the hypothesis in Lemma* 2.1, *suppose* $\deg(g, \partial B) = 0$. *Then* $|u(x)| \geqq \frac{3}{4}$ *in* B *whenever* $0 < \varepsilon \leqq \varepsilon_K$ *for some* ε_K. *In general, if* $v \in H^1(B)$ *with* $v = g$ *on* ∂B *and* $|\nabla v(x)| \leqq c/\varepsilon$, *then*

$$E_\varepsilon(v) \geqq \min\left\{E_\varepsilon(u) : u|_{\partial B} = g\right\} + C_0(K)$$

for some $C_0(K) > 0$ *provided that* $|v(0)| \leqq \frac{1}{2}$.

Proof: See proof of lemma 2 in [17].

DEFINITION 2.3. Let Ω, g be as in (1.1)–(1.3). We say a map $u : \Omega \to \mathbb{R}^2$ belongs to the class $S_g(\lambda, K)$ if

(i) $u \in H'(\Omega)$, $u = g$ on $\partial\Omega$, and $|u(x)| \leqq 1$ in Ω;

(ii) if $|u(x_0)| \leqq \frac{1}{2}$ and $x_0 \in \Omega$, then $|u(x)| \leqq \frac{3}{4}$ whenever $x \in \Omega$ and $|x - x_0| \leqq \lambda\varepsilon$;

(iii) $E_\varepsilon(u) \leqq \pi d \log\frac{1}{\varepsilon} + K$.

It is not hard to check whether $u_\varepsilon(x, t)$ is a solution of (1.1)–(1.3). Then, for any $t > 0$, $u_\varepsilon(\cdot, t) \in S_g(\lambda, K)$ provided that $E_\varepsilon(u_0) \leqq \pi d \log\frac{1}{\varepsilon} + K$ and $|\nabla u_0(x)(x)| \leqq c_0/\varepsilon$ where λ may depend on c_0, g, and Ω. We note that λ depends only on g and Ω whenever $t \geqq \varepsilon^2$. Therefore, it is useful to get some general knowledge about those maps in $S_g(\lambda, K)$.

THEOREM 2.4. *There are two positive numbers* ε_0 *and* α_0 *depending only on* λ, K, g, *and* Ω *such that, for any* $0 < \varepsilon \leqq \varepsilon_0$, $u \in S_g(\lambda, K)$, *there are* N_ε *disjoint balls* B_j *of radius* ε^{α_j}, $j = 1, \ldots, N_\varepsilon$, *with the following properties:*

(i) $\alpha_0 \leqq \alpha_j \leqq 1$ *for* $j = 1, \ldots, N_\varepsilon$ *and* $N_\varepsilon \leqq N_*(\lambda, K)$.

(ii) *The set* $\{x \in \Omega : |u(x)| \leqq \frac{1}{2}\}$ *is contained in* $\Omega \cap (\bigcup_{j=1}^{N_\varepsilon} B_j)$.

(iii) *The estimates* $\varepsilon^{\alpha_j} \int_{\partial(B_j \cap \Omega)} e_\varepsilon(u) \leqq c(\alpha_0, \lambda, K)$, $j = 1, \ldots, N_\varepsilon$, *are valid. In particular, the degrees* $d_j = \deg(u, \partial(B_j \cap \Omega))$ *are well-defined.*

(iv) *There are exactly d balls, say* $B_1, \ldots, B_d$, *such that the corresponding degrees* d_j *are not zero. If we let* $x_1, \ldots, x_d$ *be the centers of balls* $B_1, \ldots, B_d$, *then* $\min\{|x_i - x_j|, \operatorname{dist}(x_i, \partial\Omega) : i \neq j,\ i, j = 1, \ldots, d\} \geqq \delta(\lambda, K) > 0$. *Moreover, each* d_j *equals* 1 *for* $j = 1, \ldots, d$.

(v) *If* $B_j \cap \Omega \neq \emptyset$, *then if* $B_j \cap \Omega$ *is scaled by a factor of size* $\simeq \varepsilon^{\alpha_j}$, *the resulting domain is of diameter 1 and is uniformly Lipschitz (independently of* ε *and* j).

Proof: *Step 1.* Starting with a map $u \in S_g(\lambda, K)$, we are going to construct a finite sequence of maps such that each map is a simple modification of the preceding one and the final map v has the following properties:

(P1) $v \in S_g(\lambda, K)$;

(P2) the set $\{x \in \Omega : |v(x)| \leqq \frac{1}{2}\}$ is contained in d disjoint balls D_j centered at y_j and of radius ε^{β_j} for $j = 1, \dots, d$, where $\beta_j \geqq \beta(\lambda, K) > 0$ for $j = 1, \dots, d$.

(P3) the balls D_j, $j = 1, \dots d$, satisfy $\varepsilon^{\beta_j} \int_{\partial D_j} e_\varepsilon(V) \leqq c(\beta, \lambda, K)$, $\min\{|y_i - y_j|, \operatorname{dist}(y_i, \partial\Omega) : i \neq j,\ i, j = 1, \dots, d\} \geqq \delta(\lambda, K)$, and $d_j = \deg(V, \partial D_j) = 1$ for $j = 1, \dots, d$.

For this purpose we assume that ε is so small that $K \leqq \frac{1}{2} \log \frac{1}{\varepsilon}$ and thus $E_\varepsilon(u) \leqq \pi(d + \frac{1}{2}) \log \frac{1}{\varepsilon}$. Set $\alpha = 2^{-d-k_0}$ where k_0 is chosen so that $2^{-k_0+1} \leqq \frac{1}{6(d+1)}$. As in [25], for any $x \in \Omega$ there is $\beta \in [\alpha, 2\alpha]$ such that

$$\varepsilon^\beta \int_{\partial B_{\varepsilon^\beta}(x) \cap \Omega} e_\varepsilon(u) \leqq c(d)/\alpha\,. \tag{2.4}$$

In particular, $|u(x)| \geqq \frac{3}{4}$ for $x \in \partial(B_{\varepsilon^\beta}(x) \cap \Omega)$, and the degree $(u, \partial(B_{\varepsilon^\beta}(x) \cap \Omega))$ is well-defined whenever ε is sufficiently small (depending only on K, d, g, and Ω).

Let y_0 be a point in the set $\{x \in \Omega : |u(x)| \leqq \frac{1}{2}\}$. Then we choose a ball D of radius ε^β for some $\beta \in [\alpha, 2\alpha]$ and centered at y_0 so that (2.4) is valid (with y_0 in place of x). If $\deg(u, \partial(D \cap \Omega)) = 0$, then we replace u inside $D \cap \Omega$ by $\tilde{u}$, where $\tilde{u}$ minimizes the energy $\int_{D\cap\Omega} e_\varepsilon(v)dx$ with $\tilde{u} = u$ on $\partial(D \cap \Omega)$. It is not hard to see that $\varepsilon^{-\beta}(D \cap \Omega)$ is a Lipschitz domain for which we may apply Lemma 2.2 so that $|\tilde{u}(x)| \geqq \frac{3}{4}$ in $D \cap \Omega$. In this way we obtain a new map $u'(x)$ that equals $\tilde{u}$ on $D \cap \Omega$, that coincides with u on $\Omega | D$ such that

$$E_\varepsilon(u'(x)) \leqq E_\varepsilon(u)\,,$$

and that still possesses the property (ii) in the definition for the class $S_g(\lambda, K)$. Indeed, since $|u(x)| \geqq \frac{3}{4}$ on $\partial(D \cap \Omega)$, for any point $x_0 \in \Omega | D$ with $|u(x_0)| \leqq \frac{1}{2}$, the ball $\{x : |x - x_0| \leqq \lambda\varepsilon\}$ will not intersect $\partial(D \cap \Omega)$. In other words, $u'(x) \in S_g(\lambda, K)$. We then apply the same modification procedure to $u'(x)$ as we did to $u(x)$ above to obtain the second new map $u''(x)$, and so on. After finitely many repetitions of this procedure, say N_1 times, we arrive at a new map $u_1(x)$, a point $y_1 \in \{x \in \Omega : |u_1(x)| \leqq \frac{1}{2}\}$, and a ball D_1 centered at y_1 of radius ε^{β_1} for which the corresponding estimate (2.4) is valid. Moreover, $\deg(u_1, \partial(D_1 \cap \Omega)) = d_1 \neq 0$.

At this stage we will keep the ball D_1 and let $\Omega_1 = \Omega | D_1$. We apply the same arguments as above for u on Ω to u_1 on Ω_1 with $\beta \in [2\alpha, 4\alpha]$. The reason we change the range of values for β is as follows: For any $x \in \Omega_1$, let D be a ball centered at x and of radius ε^β for some $\beta \in [2\alpha, 4\alpha]$; then $D \cap \Omega_1$ is a uniformly Lipschitz domain after normalization (its Lipschitz character is independent of $\beta \in [2\alpha, 4\alpha]$, small positive ε, and $x \in \Omega_1$).

As in the previous stage, we modify u_1 on Ω_1 a total of N_2 times to find another map u_2, a point $y_2 \in \{x \in \Omega_1 : |u_2(x)| \leqq \frac{1}{2}\}$, and a ball D_2 of radius ε^{β_2}, $\beta_2 \in [2\alpha, 4\alpha]$, and centered at y_2 such that $\deg(u_2, \partial(D_2 \cap \Omega_1)) = d_2 \neq 0$. We keep the ball D_2, let $\Omega_2 = \Omega_1 | D_2$, then apply the same procedure to u_2 on Ω_2 with $\beta \in [4\alpha, 8\alpha]$, and so on.

We claim the above procedure has to stop after d iterations of modifying the maps. Indeed, if we find $d+1$ balls, $d_1, \dots, d_{d+1}$, then by Lemma 2.1 (which may apply to all domains $D_j \cap \Omega_{j-1}$, $j = 1, \dots, d+1$, $\Omega_0 = \Omega$, since all these domains become uniformly Lipschitz after a proper scaling), we have, for $j = 1, 2, \dots, d+1$, that

$$\int_{D_j \cap \Omega_{j-1}} e_\varepsilon(u_j) \geqq \pi\, |d_j| \log \frac{1}{\varepsilon}(1 - 2^j \alpha) - c(\alpha, d).$$

Therefore,

$$\begin{aligned} E_\varepsilon(u) &\geqq E_\varepsilon(u_{d+1}) \geqq \sum_{j=1}^{d+1} \pi\, |d_j| \log \frac{1}{\varepsilon}\, (1 - 2^j \alpha) - C(\alpha, d) \\ &\geqq \pi \log \frac{1}{\varepsilon} \sum_{j=1}^{d+1} (1 - 2^j 2^{-d-k_0}) \\ &\geqq \pi \left(d + \frac{2}{3}\right) \log \frac{1}{\varepsilon} - c(\alpha, d). \end{aligned} \tag{2.5}$$

This result contradicts the fact that $u \in S_g(\lambda, K)$ whenever ε is sufficiently small.

In summary, after finitely many modifications we obtain a new map v. Moreover, v satisfies both (P1) and (P2). From our constructions, it suffices to verify (P3). Since $\sum_{j=1}^d d_j = d$, the fact $d_j = 1$ follows from (2.5). To show that the centers of the balls D_j are distinct and lie strictly inside Ω, we consider a new map $\tilde{v}$ such $\tilde{v} = v$ on $D_j \cap \Omega$, $j = 1, \dots, d$, and $\tilde{v}$ minimizes

$$\int_{\Omega | \cup_{j=1}^d D_j} e_\varepsilon(u) dx$$

with $\tilde{v} = v$ on $\partial(\Omega | \cup_{j=1}^d D_j)$. Then, the arguments in the proof of theorem A of [17] show that

$$E_\varepsilon(\tilde{v}) \geqq \pi d \log \frac{1}{\rho} + dI(\varepsilon, \rho) + O(\rho) + W_g(\tilde{y}), \tag{2.6}$$

whenever ε is suitably small and where $\rho \geqq \rho(\varepsilon)$, $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_d)$, and $y = (y_1, \dots, y_d)$ satisfy $|\tilde{y} - y| \leqq \rho^2$ since $W_g(\tilde{y}) \to +\infty$ whenever two points $\tilde{y}_i$ and $\tilde{y}_j (i \neq j)$ coalesce or one of the point $\tilde{y}_i$ tends to $\partial\Omega$. Thus we conclude that (P3) is also true. Note that the value (for all sufficiently small ε) of $\min\{|y_i - y_j|, \operatorname{dist}(y_i, \partial\Omega) : i \neq j,\ i, j = 1, \dots, d\}$ is bounded below by a positive constant depending only on K, g, and Ω.

Step 2. We want to show that the total number of modifications $N = \sum_{j=1}^{d+1} N_j$ that take place in constructing the map v with properties (P1), (P2), and (P3) is uniformly bounded by a constant depending only on λ, K, g, and Ω. Here N_{d+1} is the number of modifications that have to be made to cover the set $\{x \in \Omega \mid \bigcup_{j=1}^{d} D_j : |U_d(x)| \leqq \frac{1}{2}\}$ so that the resulting map v has the property (P2).

To show the above fact, we use (2.3) to obtain first that

$$E_\varepsilon(u) - E_\varepsilon(v) \leqq K + C(g, \Omega).$$

On the other hand, all maps in this modification procedure are in class $S_g(\lambda, K)$. We employ the proof of Lemma 2.2 (cf. [17], lemma 2) to obtain that each modification decreases the total energy of the map by at least $\frac{\lambda^2}{16}$ whenever ε is small enough (depending only on λ, K, g, and Ω). Thus $N \cdot \frac{\lambda^2}{16} \leqq K + C(g, \Omega)$, that is,

$$N \leqq N(\lambda, K).$$

Therefore, the values of the maps v and u can be different only on a union of at most N balls of sizes $\leqq \varepsilon^\alpha$. In other words, the set $\{x \in \Omega : |u(x)| \leqq \frac{1}{2}\}$ is contained in a union of at most $N + d$ balls of sizes $\leqq \varepsilon^\alpha$.

Step 3. We can now complete the proof of Theorem 2.4. Since the set $\{x \in \Omega : |u(x)| \leqq \frac{1}{2}\}$ is contained in a union of at most $N + d$ balls of size $\leqq \varepsilon^\alpha$, we want to employ the grouping and induction argument as in [25] and theorem 2.8 in [12] to these at most $N + d$ balls of size $\leqq \varepsilon^\alpha$. As a result, we want to find N_ε balls $\overline{B}_j$ of radius $\varepsilon^{\overline{\alpha}_j}$, $j = 1, \dots, N_\varepsilon$, with the following properties whenever $\varepsilon \leqq \varepsilon_0$:

(I) $\overline{\alpha}_j \in [\alpha_0, \alpha]$ for $j = 1, \dots, N_\varepsilon$ and $N_\varepsilon \leqq N + d \leqq N_*(\lambda, K)$. Here α_0 is a positive constant that may depend on N_*.

(II) The set $\{x \in \Omega : |u(x)| \leqq \frac{1}{2}\}$ is contained in $\Omega \cap \bigcup_{j=1}^{N_\varepsilon} \overline{B}_j$, and the ball $\varepsilon^{-\overline{\alpha}_j/3} B_j$ (scale $\overline{B}_j$ by a factor $\varepsilon^{-\overline{\alpha}_j/3}$ about its center) are pairwise disjoint for $j = 1, 2, \dots, N_\varepsilon$.

To prove properties (I) and (II), we need the following:

LEMMA 2.5. *Let $B_1, B_2, \dots, B_N$ be N balls in $\mathbb{R}^2$ with radii not larger than ε^α for some $\alpha \in (0, \frac{1}{4})$ and for $j = 1, \dots, N$. Then there are a positive number α_0 (depending only on α and N) and balls $\overline{B}_j$ of radius $\varepsilon^{\overline{\alpha}_j}$ for $j = 1, \dots, N_\varepsilon \leqq N$ such that properties* (I) *and* (II) *are valid provided that ε is sufficiently small.*

Proof: Let $A = \bigcup_{j=1}^{N} B_j$. We are going to prove this lemma on covering by induction on the number of connected components of A. If A is connected, then we simply take $\overline{\alpha}_1 = \frac{\alpha}{3}$ and a ball $\overline{B}_1$ of radius $\varepsilon^{\overline{\alpha}_1} \geqq 2N\varepsilon^\alpha$ (this inequality will be valid whenever ε is suitably small) such that $A \subset \overline{B}_1$. The conclusion of the lemma follows automatically.

Suppose that the conclusions of Lemma 2.5 are true whenever (the number of connected components of A) $\leqq k \leqq N-1$. Moreover, these $\overline{\alpha}_j$'s satisfy $\overline{\alpha}_j \geqq \frac{\alpha}{3^k}$ for each j. We want to show that Lemma 2.5 is true when the number of the connected components of A is $k+1 \leqq N$, and that each $\overline{\alpha}_j$ in the lemma can be chosen to be not less than $\frac{\alpha}{3^{k+1}}$ whenever ε is small enough.

For this purpose, we let $A_1, \ldots, A_{k+1}$ be connected components of A. Without loss of generality, we may assume that the diameter of A is larger than $3(k+1)\varepsilon^{\frac{2\alpha}{3^{k+1}}}$, for otherwise we may simply choose a ball B of radius $\leqq \varepsilon^{\frac{\alpha}{3^{k+1}}}$ that covers A entirely (whenever ε is small enough), and then the conclusion of the covering lemma is obvious.

Now we let $x', x'' \in A$ be such that $|x'-x''| = \operatorname{diam} A \geqq 3(k+1)\varepsilon^{\frac{2\alpha}{3^{k+1}}}$. We may find a $\rho_0 \in (0, 3(k+1)\varepsilon^{\frac{2\alpha}{3^{k+1}}})$ such that the boundary $\partial B_r(x')$ of the ball $B_r(x')$ will not intersect any of the A_j's for $j = 1, \ldots, k+1$ and for any $r \in [\rho_0 - \varepsilon^{\frac{2\alpha}{3^{k+1}}}, \rho_0 + \varepsilon^{\frac{2\alpha}{3^{k+1}}}]$. Then it is obvious that $A \cap B_{\rho_0}(x') = A'$, and $A'' = A \sim A'$ contains some of each $A_1, \ldots, A_{k+1}$. We may apply the induction step to both A' and A'' to conclude that $A = A' \cup A''$ can be covered by balls $\overline{B}_j$ of radius $\varepsilon^{\overline{\alpha}_j}$, $\overline{\alpha}_j \geqq \frac{\alpha}{3^k}$. Now since $\operatorname{dist}(A', \partial B_{\rho_0}(x')) \geqq \varepsilon^{\frac{2\alpha}{3^{k+1}}}$, and since $\operatorname{dist}(A'', \partial B_{\rho_0}(x') \geqq \varepsilon^{\frac{2\alpha}{3^{k+1}}}$, the conclusions of the covering lemma follow. This completes the induction argument.

Now we can apply Fubini's theorem to find balls B_j (having the same center as $\overline{B}_j$) of radius ε^{α_j}, $\alpha_j \in [\overline{\alpha}_j/3, \overline{\alpha}_j]$, such that (i), (ii), and (iii) of Theorem 2.4 are valid. Parts (iv) and (v) follow from the same proof as in Step 1.

COROLLARY 2.6. *Let $u \in S_g(\lambda, K)$ and B_j, $j = 1, \ldots, N_\varepsilon, 0 < \varepsilon \leqq \varepsilon_0$, be as in Theorem 2.4. Suppose that $\overline{x}_j$ is the center of B_j and that* $\deg(u, \partial B_j) = 1$ *for $j = 1, \ldots, d$. Then $N_\varepsilon = d$ whenever*

$$E_\varepsilon(u) \leqq \pi d \log \frac{1}{\varepsilon} + W_g(\overline{x}) + c_0$$

where $c_0 = d\gamma + \frac{\lambda^2}{32}$, and γ is the value defined in theorem IX.3 of [2].

Proof: Suppose $N_\varepsilon > d$. Then we replace values of u in B_j, $j = d+1, \ldots, N_\varepsilon$, by its corresponding values of minimizers of the energy functional on B_j (with the Dirichlet boundary condition given by u). The resulting map v has the similar property (2.6) as for $\widetilde{v}$.

By theorem IV.3 of [2], $I(\varepsilon, \rho) = \pi d \log \frac{\rho}{\varepsilon} + d\gamma + o(1)$ as $\varepsilon \to 0^+$ whenever $\rho \geqq \rho(\varepsilon)$. By choosing ρ small, we have

$$E_\varepsilon(v) \geqq \pi d \log \frac{1}{\varepsilon} + \gamma d + W_g(\overline{x}) + o(1) .$$

Here $o(1)$ is a quantity that goes to zero when $\varepsilon \to 0^+$. Since $E_\varepsilon(u) \geqq E_\varepsilon(v) + \frac{\lambda^2}{16}$ (see Step 2), we obtain a contradiction to the assumption on the boundedness of $E_\varepsilon(u)$.

COROLLARY 2.7. *Let $u \in S_g(\lambda, K)$ be as in Corollary 2.6 and*

$$E_\varepsilon(u) \leqq \pi d \log \frac{1}{\varepsilon} + d\gamma + W_g(\bar{x}) + c_0 .$$

Then

$$\int_{\Omega_\varepsilon} \frac{(1 - |u|^2)^2}{\varepsilon^2} dx \leqq C \tag{2.7}$$

where C is a constant depending on λ and K and where $\Omega_\varepsilon = \Omega | \cup_{j=1}^d B_j$.

Proof: By Corollary 2.6, $|u(x)| \geqq \frac{1}{2}$ on Ω_ε. By Lemma 2.1

$$\int_{\bigcup_{j=1}^d B_j} e_\varepsilon(u) dx \geqq \sum_{j=1}^d \pi \log \frac{1}{\varepsilon}(1 - \alpha_j) - c .$$

On the other hand, by Theorem 2.7(iv) and proposition 3.4 of [26], we have

$$\int_{\Omega_\varepsilon} \frac{1}{2} |\nabla u|^2 dx \geqq \sum_{j=1}^d \pi \log \frac{\delta(\lambda, K)}{\varepsilon^{\alpha_j}} - C .$$

Therefore, as $u \in S_g(\lambda, K)$, we have

$$\frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon} (1 - |u|^2) dx \leqq \text{const}$$

where the constant depends only on λ, K, g, and Ω.

3. The Lower Bound of the Mobilities

Let us consider first the finite time behavior of solutions $u_\varepsilon(x, t)$ of (1.1)–(1.3). The initial data u_0 are assumed to satisfy the following:

ASSUMPTION 3.1.
(i) *u_0 is smooth with $|u_0(x)| \leqq 1$ in Ω;*
(ii) *$E_\varepsilon(u_0) \leqq \pi d \log \frac{1}{\varepsilon} + K_1$ for a constant K_1; and*
(iii) *$\int_\Omega \rho^2(x) e_\varepsilon(u_0) dx \leqq K_2$ for a constant K_2 where $\rho(x) = \operatorname{dist}(x, \{b_1, \dots, b_d\})$, and $b_1, \dots, b_d$ are d distinct points in Ω.*

Under Assumption 3.1 and our hypothesis on g and Ω, the global existence of the unique smooth solution $u_\varepsilon(x, t)$ of (1.1)–(1.3) can be shown by a rather standard method (cf. [1]). Moreover, we have

$$\frac{d}{dt} E_\varepsilon(u_\varepsilon(\cdot, t)) = - \int_\Omega \left| \frac{\partial}{\partial t} u_\varepsilon(x, t) \right|^2 dx ,$$

and thus

$$\sup_{t>0}\int_0^t\int_\Omega\left|\frac{\partial}{\partial t}u_\varepsilon(x,\tau)\right|^2 dx\,d\tau + E_\varepsilon(u_\varepsilon(\cdot,t)) \leqq E_\varepsilon(u_0)\leqq \pi d\log\frac{1}{\varepsilon}+K. \tag{3.1}$$

By using a scaling argument and usual parabolic estimates, it is also easy to see that

$$|\nabla u_\varepsilon(x,t)|^2+\left|\frac{\partial}{\partial t}u_\varepsilon(x,t)\right|\leqq\frac{c_*}{\varepsilon^2}, \tag{3.2}$$

for a constant c_* depending only on g and Ω and for all $t\geqq\varepsilon^2$. The estimate (3.2) will be also true for $0<t<\varepsilon^2$ provided that $u_0^\varepsilon(x)=u_0(\varepsilon x)$ is such that

$$\|\nabla u_0^\varepsilon(x)\|_{L^\infty}+\sup_{x,y}\frac{|\nabla u_0^\varepsilon(x)-\nabla u_0^\varepsilon(y)|}{|x-y|^\alpha}\leqq K_3, \tag{3.3}$$

for some constants α and K_3. In this case, the constant c_* in (3.2) will, of course, also depend on K_3 and α for $0<t\leqq\varepsilon^2$.

(3.1) and (3.2) imply that $u_\varepsilon(x,t)\in S_g(\lambda,K)$ for constant λ. Also, by (2.3), we have

$$E_\varepsilon(u_\varepsilon(\cdot,t))\geqq\pi d\log\frac{1}{\varepsilon}-C_1,$$

and hence

$$\int_0^\infty\int_\Omega\left|\frac{\partial}{\partial t}u_\varepsilon(x,t)\right|^2 dx\,dt\leqq C_1+K_1. \tag{3.4}$$

Next, we calculate

$$\frac{d}{dt}\int_\Omega\rho^2(x)e_\varepsilon(u_\varepsilon(x,t))\,dx\leqq\frac{1}{2}\int_\Omega\rho^2(x)|\nabla u_\varepsilon|\,dx+2\int_\Omega\left|\frac{\partial}{\partial t}u_\varepsilon\right|^2dx. \tag{3.5}$$

Here we have used the fact that $|\nabla\rho(x)|\leqq 1$ in Ω. Therefore,

$$\int_\Omega\rho^2(x)e_\varepsilon(u_\varepsilon(x,t))\,dx\leqq 2e^t\int_0^t\int_\Omega\left|\frac{\partial}{\partial t}u_\varepsilon\right|^2dx\,dt+K_2. \tag{3.6}$$

For any $T\in(0,\infty)$ and any $\delta\in(0,\delta_0)$ where

$$2\delta_0=\min\left\{|b_i-b_j|,\operatorname{dist}(b_i,\partial\Omega): i\neq j,\ i,j=1,\dots,d\right\},$$

we let $\Omega_\delta = \Omega \setminus \bigcup_{j=1}^{d} B_\delta(b_j)$ and $Q_{\delta,T} = \Omega_\delta \times [0,T]$. From (3.4) and (3.6), we have $u_\varepsilon \in H'(Q_{\delta,T})$ with

$$\int_{\Omega_\delta} e_\varepsilon(u_\varepsilon(x,t))\,dx \leqq c(\delta,K_1,K_2)e^T, \quad \forall t \in [0,T]. \tag{3.7}$$

To study the asymptotic behavior of solutions u_ε of (1.1)–(1.3) as $\varepsilon \to 0^+$, we need to make the following additional assumption on u_0 (which, in general, may depend on ε):

ASSUMPTION 3.2. *The initial data $u_0^\varepsilon(x)$ converge to*

$$\prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{ih_0(x)}$$

as $\varepsilon \to 0^+$ where $h_0(x)$ is a function in $H'(\Omega)$.

Now, for any sequence of $\varepsilon_n \downarrow 0$, there is a subsequence (still denoted by ε_n) so that $u_{\varepsilon_n}(x,t) \to u_0(x,t)$ weakly in $H'_{\text{loc}}(\overline{\Omega}\setminus\{b_1,\dots,b_d\} \times \mathbb{R}_+)$ and strongly in $L^2_{\text{loc}}(\overline{\Omega} \times \mathbb{R}_+)$. The latter is because $|u_\varepsilon(x,t)| \leqq 1$. Moreover, $|u_0(x,t)| = 1$ are a.e. in $\Omega \times \mathbb{R}_+$.

Since $u_\varepsilon \wedge \frac{\partial}{\partial t} u_\varepsilon = \operatorname{div}(u_\varepsilon \wedge \nabla u_\varepsilon)$ in $\Omega \times \mathbb{R}_+$, we deduce that (via $|u_0|(x,t) = 1$ a.e.)

$$\begin{cases} \dfrac{\partial u_0}{\partial t} = \Delta u_0 + |\nabla u_0|^2 u_0 \text{ in } \Omega/\{b_1,\dots,b_d\} \times \mathbb{R}_+, \\ u_0(x,0) = \displaystyle\prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{ih_0(x)}, \\ u_0 = g \quad \text{on } \partial\Omega \times \mathbb{R}_+. \end{cases} \tag{3.8}$$

Since the images of $u_0(x,t)$ lie in the unit circle, the function $u_0(x,t)$ is smooth in $\overline{\Omega}\setminus\{b_1,\dots,b_d\} \times \mathbb{R}_+$ (cf. [17]). Moreover, by (3.4) and (3.7), we can deduce that, for any $t > 0, 0 < \delta \leqq \delta_0$, the degrees $\deg(u_0(\cdot,t), \partial B_\rho(b_j)), j = 1,\dots,d$, are well-defined and all equal to 1 by Assumption 3.2. Therefore, we may write

$$u_0(x,t) = \prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{ih_0(x,t)} \tag{3.9}$$

It is obvious that

$$\begin{cases} \dfrac{\partial}{\partial t} h_0(x,t) = \Delta h_0(x,t) \quad \text{in } \Omega\setminus\{b_1,\dots,b_d\} \times \mathbb{R}_+, \\ h_0(x,t) = h_0(x) \quad \text{on } \partial\Omega \times \mathbb{R}_+, \\ h_0(x,0) = h_0(x). \end{cases} \tag{3.10}$$

At this stage, we do not know if $h_0(x,t)$ satisfies the heat equation in $\Omega \times \mathbb{R}_+$. We also do not know if such $h_0(x,t)$ is determined by the limit of the whole family $u_\varepsilon(x,t)$ instead of a special subsequence $\{u_{\varepsilon_n}(x,t)\}$.

THEOREM 3.3. *The function $h_0(x,t)$ in (3.10) satisfies*

$$\sup_{t>0}\left[\|\nabla h_0(x,t)\|_{L^2}^2\right] + \int_0^t \int_\Omega \left|\frac{\partial}{\partial t}h_0(x,t)\right|^2 dx\,dt \leqq C,$$

and C depends only on g, Ω, and $K = \max(K_1, K_2, K_3)$. In particular, $h_0(x,t)$ satisfies the heat equation in $\Omega \times \mathbb{R}_+$. Consequently, we have

$$u_\varepsilon(x,t) \to \prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{ih_0(x,t)}$$

in $L^2_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}_+)$ and weakly in

$$H'_{\mathrm{loc}}\left(\overline{\Omega}\backslash\{b_1,\dots,b_d\} \times \mathbb{R}_+\right)$$

when $\varepsilon \to 0$.

Proof: Let us consider a function $u \in S_g(\lambda, k)$, and let $B_1,\dots,B_{N_\varepsilon}$ be balls given in Theorem 2.4. Let $x_1^\varepsilon,\dots,x_d^\varepsilon$ be centers of $B_1,\dots,B_d$, respectively; then

$$u(x) = \prod_{j=1}^{d} \frac{x-x_j^\varepsilon}{|x-x_j^\varepsilon|} e^{ih_\varepsilon(x)} \cdot \rho_\varepsilon(x) \quad \text{for all } x \in \Omega\backslash \bigcup_{j=1}^{N_\varepsilon} = \Omega_\varepsilon \tag{3.11}$$

where $h_\varepsilon(x)$ is a well-defined function, single-valued in Ω_ε, and

$$\frac{1}{2} \leqq \rho_\varepsilon(x) \leqq 1, \quad x \in \Omega_\varepsilon.$$

Such h_ε is uniquely determined if $h_\varepsilon(x_0) \in [0, 2\pi)$ for some given $x_0 \in \partial\Omega$. We note that (3.11) is true because

$$\deg(u, \partial B_j) = 0 \quad \text{for } j = d+1,\dots,N_\varepsilon.$$

We have, on the one hand,

$$\int_{\Omega_\varepsilon} e_\varepsilon(u)dx \leqq \pi \sum_{j=1}^{d} \alpha_j \log\frac{1}{\varepsilon} + C(\lambda, K).$$

On the other hand,

$$\int_{\Omega_\varepsilon} e_\varepsilon(u)dx \geqq \int_{\Omega_\varepsilon} \rho_\varepsilon^2|\nabla\Theta|^2 + \rho_\varepsilon^2|\nabla h_\varepsilon|^2 + 2\rho_\varepsilon^2\nabla\Theta_\varepsilon \cdot \nabla h_\varepsilon$$

where Θ_ε is a multivalued-harmonic function on Ω_ε so that

$$e^{i\Theta_\varepsilon} = \prod_{j=1}^{d} \frac{x - x_j^\varepsilon}{|x - x_j|^\varepsilon}\,.$$

Since $|\nabla\Theta_\varepsilon|(x) \leqq \varepsilon^{-\alpha}$ for $x \in \Omega_\varepsilon$, we have

$$\begin{aligned}\int_{\Omega_\varepsilon} |\rho_\varepsilon^2 - 1|\ |\nabla\Theta_\varepsilon|^2 &\leqq \varepsilon^{-2\alpha}\left(\int_{\Omega_\varepsilon} (\rho_\varepsilon^2 - 1)^2\,dx\right)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \\ &\leqq C(K,\Omega)\varepsilon^{-2\alpha+1}\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{2}}.\end{aligned}$$

Note $\alpha < \frac{1}{2}$ (see Step 1 of the proof to Theorem 2.4). Since

$$\int_{\Omega_\varepsilon} |\nabla\Theta_\varepsilon|^2\,dx \geqq \pi\sum_{j=1}^{d}\alpha_j \log\frac{1}{\varepsilon} - C(\lambda, K),$$

we have

$$\int_{\Omega_\varepsilon} \rho_\varepsilon^2\,|\nabla h_\varepsilon|^2 + 2\rho_\varepsilon^2\nabla\Theta_\varepsilon\cdot\nabla h_\varepsilon \leqq C(\lambda, K).$$

Similarly,

$$\int_{\Omega_\varepsilon} |\rho_\varepsilon^2 - 1|\ |\nabla\Theta_\varepsilon|\ |\nabla h_\varepsilon| \leqq o(1)\,\|\nabla h_\varepsilon\|_{L^2(\Omega_\varepsilon)}\,.$$

To show $\int_{\Omega_\varepsilon} |\nabla h_\varepsilon|^2 \leqq C(\lambda, K)$, it suffices to verify

$$\int_{\Omega_\varepsilon} \nabla\Theta_\varepsilon\nabla h_\varepsilon\,dx$$

is bounded by a constant. We calculate

$$\int_{\Omega_\varepsilon} \nabla\Theta_\varepsilon\cdot\nabla h_\varepsilon = \int_{\partial\Omega} h_\varepsilon\frac{\partial\Theta_\varepsilon}{\partial\nu} + \sum_{j=1}^{N_\varepsilon}\int_{\partial B_j} (h_\varepsilon - \overline{h}_\varepsilon)\,\frac{\partial\Theta_\varepsilon}{\partial\nu} \leqq C(\lambda, K).$$

Here $\overline{h}_\varepsilon = ⨍_{\partial B_j} h_\varepsilon$. Note we have used the facts that $\int_{\partial B_j}\frac{\partial\Theta_\varepsilon}{\partial\nu} = 0$ and $|h_\varepsilon - \overline{h}_\varepsilon| \leqq C(\lambda, K)$ on each ∂B_j, $j = 1, \ldots, N_\varepsilon$.

We then apply the above arguments to each $u_\varepsilon(x,t) \in S_g(\lambda, K), t > 0$. From (3.7), we see $x_j^\varepsilon \to b_j$ as $\varepsilon \to 0$ for $j = 1, \ldots, d$. Since, for a.e. $t, u_{\varepsilon_n}(x,t) \to \prod_{j=1}^{d}\frac{x-b_j}{|x-b_j|}e^{ih_0(x,t)}$, we see $h_0(x,t)$ is a weak limit of $h_{\varepsilon_n}(x,t)$ in $H'(\Omega)$. Here to avoid the trivial ambiguity, we also assume $h_0(x_0) \in (0, 2\pi)$ for the given $x_0 \in \partial\Omega$. Thus, $\int_\Omega |\nabla h_0(x,t)|^2\,dx \leqq C(\lambda, K)$ for all $t > 0$. From the latter fact one easily shows that $h_0(x,t)$ satisfies the heat equation in $\Omega \times \mathbb{R}_+$. The conclusion of Theorem 3.3 follows.

Remark 3.4. Since

$$u_{\varepsilon_n}(x,t) \to \prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{i\, h_0(x,t)}$$

in $L^2_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}_+)$ and weakly in $H'_{\mathrm{loc}}(\overline{\Omega}|\{b_1,\ldots,b_d\} \times \mathbb{R}_+)$, we see that

$$\int_0^T \int_\Omega \left|\frac{\partial h_0}{\partial t}\right|^2 (x,t)\,dx\,dt \leqq \lim \int_0^T \int_\Omega \left|\frac{\partial u_{\varepsilon_n}}{\partial t}\right|^2 dx\,dt \leqq C(K).$$

COROLLARY 3.5. *Suppose the initial data* u_0 *in* (1.3) *satisfies Assumptions* 3.1 *and* 3.2 *with* $K_1 \leqq W_g(b) + c_0$ *where* $b = (b_1,\ldots,b_d)$ *and* c_0 *is as defined in Corollary* 2.6. *Then, for any* $T > 0$, *the sets* $G(t) = \{x \in \Omega : |u_\varepsilon(x,t)| \leqq \frac{1}{2}\}, t \in (0,T]$, *converge uniformly in* t *to* $\{b_1,\ldots,b_d\}$ *in the Hausdorff distance as* $\varepsilon \to 0$.

Proof: The proof is an easy consequence of Corollary 2.6, the estimate (3.7), and Lemma 2.1.

Remark 3.6. Corollary 3.5 shows that with some suitable initial data, there are no new essential vortices created in a finite time outside any neighborhood of the initial vortices whenever ε is sufficient small. (Compare with the result of [22], section 4).

Let us now consider the following scaled equations:

$$(3.12) \qquad \begin{cases} \frac{1}{\lambda_\varepsilon}\frac{\partial}{\partial t} u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2}\left(1 - |u_\varepsilon|^2\right) u_\varepsilon & \text{in } \Omega \times \mathbb{R}_+\,, \\ u_\varepsilon(x,t) = g(x) & \partial\Omega \times \mathbb{R}_+\,, \\ u_\varepsilon(x,0) = u_0^\varepsilon(x) & \end{cases}$$

Here we assume that

$$(3.13) \qquad \lim_{\varepsilon\to 0} \lambda_\varepsilon = \infty \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\lambda_\varepsilon}{\log \varepsilon} = 0\,.$$

THEOREM 3.7. *Suppose the initial data satisfy Assumptions* 3.1 *and* 3.2. *Then as* $\varepsilon \to 0$, *solutions* $u_\varepsilon(x,t)$ *of* (3.12) *verify:*

$$u_\varepsilon(x,t) \to \prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{i\, h(x)} \quad \text{in } L^2_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}_+)\,,$$

where $\Delta h(x) = 0$ in Ω and $h(x) = h_0(x)$ on $\partial\Omega$.

Proof: As for (3.1), one has

$$\sup_{t>0} \frac{1}{\lambda_\varepsilon} \int_0^t \int_\Omega \left| \frac{\partial}{\partial t} u_\varepsilon \right|^2 dx\,dt + E_\varepsilon(u_\varepsilon)(t) \leqq E_\varepsilon\left(u_0^\varepsilon\right) \leqq \pi d \log \frac{1}{\varepsilon} + K_1 . \tag{3.14}$$

Also, as for (3.5), one gets

$$\frac{d}{dt} \int_\Omega \rho^2(x) e_\varepsilon\left(u_\varepsilon(x,t)\right) dx \leqq \int_\Omega^\Omega \rho^2(x) e_\varepsilon\left(u_\varepsilon(x,t)\right) dx + 2\lambda_\varepsilon \cdot \frac{1}{\lambda_\varepsilon} \int_\Omega \left| \frac{\partial}{\partial t} u_\varepsilon \right|^2 dx . \tag{3.15}$$

Therefore,

$$\int_\Omega \rho^2(x) e_\varepsilon(u_\varepsilon(x,t)) dx \leqq 2\lambda_\varepsilon e^t \int_0^t \frac{1}{\lambda_\varepsilon} \int_\Omega \left| \frac{\partial}{\partial t} u_\varepsilon \right|^2 dx\,dt + K_2 \leqq C(K) \lambda_\varepsilon e^t . \tag{3.16}$$

Note also that $u_\varepsilon(\cdot, t) \in S_g(\lambda, K)$ for any $t > 0$ whenever ε is sufficiently small. We claim the following: For any $t > 0$ and any sequences of $\varepsilon_n \downarrow 0$, there is a subsequence of $u_{\varepsilon_n}(x,t)$ that we will still denote by u_{ε_n} such that

$$u_{\varepsilon_n}(x,t) \to \prod_{j=1}^d \frac{x - a_j}{|x - a_j|} e^{ih_a(x)}, \quad x \in \Omega ,$$

in $L^2(\Omega)$ and weakly in $H'_{\text{loc}}(\overline{\Omega} | \{a_1, \ldots, a_d\})$ where $a_1, \ldots, a_d$ are d distinct points in Ω. Moreover, $h_a(x) \in H'(\Omega)$. This claim shall be proved in the next section (see Lemma 4.1).

Given the estimate (3.16) and the claim above, we get $a_j = b_j$ for $j = 1, \ldots, d$. Since

$$\lambda_{\varepsilon_n} \int_0^\infty \int_\Omega \left| \Delta u_{\varepsilon_n} + \frac{1}{\varepsilon_n^2} u_{\varepsilon_n} (1 - |u_{\varepsilon_n}|^2) \right|^2 dx\,dt \leqq C(K) ,$$

we see that there is a subsequence (still denoted by ε_n) such that

$$\left\| \Delta u_{\varepsilon_n} + \frac{1}{\varepsilon_n^2} u_{\varepsilon_n} (1 - |u_{\varepsilon_n}|^2) \right\|_{L^2(\Omega)} (t) \to 0 \quad \text{for a.e. } t > 0 .$$

This, together with the claim above, yields $h_a(x) = h(x), \Delta h(x) = 0$ (cf. (3.8)).

To summarize, we have shown that, for any sequence of $\varepsilon_n \downarrow 0$, there is a subsequence of $\{u_{\varepsilon_n}\}$ such that, for almost all $t > 0$,

$$u_{\varepsilon_n}(x,t) \to \prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{ih(x)} \quad \text{in } L^2(\Omega),\ \Delta h(x) = 0,$$

In particular,

$$\int_0^T \left\| u_{\varepsilon_n}(x,t) - \prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{ih(x)} \right\|^2_{L^2(\Omega)} dt \to 0$$

as $\varepsilon_n \downarrow 0$, for any T. The latter follows from Egorov's theorem and the fact that

$$\left\| u_{\varepsilon_n}(x,t) - \prod_{j=1}^{d} \frac{x-b_j}{|x-b_j|} e^{i\,h(x)} \right\|_{L^2(\Omega)} \leqq C(\Omega), \quad \forall t > 0.$$

Since $\varepsilon_n \downarrow 0$ is arbitrary, the conclusion of Theorem 3.7 follows.

Remark 3.8. Theorem 3.7 shows that the mobilities of vortices cannot be much smaller than a multiple of $\log \frac{1}{\varepsilon}$.

Remark 3.9. If we replace λ_ε in (3.12) by $\delta \log \frac{1}{\varepsilon}$ for a small positive number δ, then it follows from the proof of Theorem 3.7 that for any $\varepsilon_n \downarrow 0$, there is a subsequence of $u_{\varepsilon_n}(x,t)$ such that

$$\left\| \Delta u_{\varepsilon_n} + \frac{1}{\varepsilon_n^2} u_{\varepsilon_n} \left(1 - |u_{\varepsilon_n}|^2\right) \right\|_{L^2(\Omega)} (t) \to 0 \quad \text{for a.e. } t.$$

Moreover, for any $t > 0$, there is a subsequence of $u_{\varepsilon_n}(x,t)$ that converges to $\prod_{j=1}^{d} \frac{x-a_j}{|x-a_j|} e^{ih_a(x)}$ in $L^2(\Omega)$ and weakly in $H'_{\text{loc}}(\overline{\Omega}\{a_1,\dots,a_d\})$ for some $a \in \Omega^d$ with $|b-a| \leqq \eta(\delta,K)$ where $\eta(\delta,K) \to 0$ when $\delta \to 0$. Finally, for a.e. t, $\Delta h_a(x) = 0$ in Ω.

4. The Upper Bound on the Mobilities

For a steady state solution $u_\varepsilon(x)$ of (1.1)–(1.2) with $E_\varepsilon(u_\varepsilon) \leqq \pi d \log \frac{1}{\varepsilon} + K$, chapter X in [2] implies the following: For any sequence of $\varepsilon_n \downarrow 0$, there is a subsequence of $\{u_{\varepsilon_n}\}$ that converges to a map of the form

$$\prod_{j=1}^{d} \frac{x-a_j}{|x-a_j|} e^{i\,h(x)} \quad \text{in } L^2(\Omega) \cap H'_{\text{loc}}\left(\overline{\Omega} | \{a_1,\dots,a_d\}\right),$$

for some d distinct points $a_1,\dots,a_d$ in Ω. Moreover, $\Delta h = 0$ in Ω, and the point $a = (a_1,\dots,a_d)$ is a critical point of the renormalized energy $W_g(\cdot)$ in $\overline{\Omega}^d$. If in addition $u_\varepsilon(x)$ minimizes (1.4), then by a theorem of [1], we can conclude that for a solution $u_\varepsilon(x,t)$ of (1.1)–(1.3) with the initial data u_0 satisfying Assumption 3.1, there is a time $T(\varepsilon, u_0, g, \Omega)$ such that $u_\varepsilon(x,t)$ has exactly d distinct zeros of degree 1 whenever $t \geqq T(\varepsilon, u_0, g, \Omega)$. Moreover, if $a_1^\varepsilon(t), \dots, a_d^\varepsilon(t)$ are zeros of $u_\varepsilon(x,t)$, then all $a_j^\varepsilon(t) \in C'(T,\infty)$ and $\lim_{t\to\infty} a_j^\varepsilon(t) = a_j^\varepsilon$ exists for $j = 1, \dots, d$.

The aim of this section is to establish various facts similar to the above-stated theorem of [1] for all times $t \geqq T_\varepsilon$ where T_ε is chosen so that

$$\lim_{\varepsilon\to 0} \frac{T_\varepsilon}{\log \frac{1}{\varepsilon}} = \infty . \tag{4.1}$$

For this reason, we consider the following

$$\begin{cases} \frac{1}{T_\varepsilon}\frac{\partial}{\partial t} u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{in } \Omega \times \mathbb{R}_+ \\ u_\varepsilon(x,t) = g(x) & \text{on } \partial\Omega \\ u_\varepsilon(x,0) = u_0^\varepsilon(x) \end{cases} \tag{4.2}$$

where T_ε satisfies (4.1) and where u_0^ε satisfies Assumptions 3.1 and 3.2.

LEMMA 4.1. (GENERAL CONVERGENCE THEOREM) *Let $u_\varepsilon \in S_g(\lambda, K)$. Then, for any sequence of $\varepsilon_n \downarrow 0$, there is a subsequence of $\{u_{\varepsilon_n}\}$ that converges to a map of form*

$$\prod_{j=1}^{d} \frac{x - b_j}{|x - b_j|} e^{ih(x)} \quad \textit{in } L^2(\Omega) \textit{ and weakly in } H'_{\mathrm{loc}}(\overline{\Omega} | \{b_1, \dots, b_d\})$$

where $b_1, \dots, b_d$ are d distinct points in Ω and $h(x) \in H'(\Omega)$.

Proof: For $u_\varepsilon \in S_g(\lambda, K)$, we let B_j, $j = 1, \dots, N_\varepsilon$, be balls in Theorem 2.4. We replace u_ε on each B_j by $\widetilde{U}_\varepsilon$, which minimizes $\int_{B_j} e_\varepsilon(V)dx$ with $V = u_\varepsilon$ on ∂B_j for $j = d+1, \dots, N_\varepsilon$. Thus, in particular, $|\widetilde{U}_\varepsilon| \geqq \frac{3}{4}$ on each B_j, $j = d+1, \dots, N_\varepsilon$. We denote the resulting map $\widetilde{U}_\varepsilon$.

Let $\delta \in (\varepsilon^{\alpha_0}, \delta(\lambda, K))$ where $\alpha_0, \delta(\lambda, K)$ are given in Theorem 2.4, and suppose that $\bigcup_{j=1}^{d} \partial B_\delta(x_j)$ does not intersect with $\bigcup_{j=d+1}^{N_\varepsilon} B_j$. Except for a set of $\delta \in (\varepsilon^{\alpha_0}, \delta(\lambda, K))$ whose measure $\leqq N_*(\lambda, K)\varepsilon^{\alpha_0}$, the latter assumption is valid.

For such a δ, we have

$$\begin{aligned} \pi d \log \frac{\delta}{\varepsilon} - C(\lambda, K) &\leqq \sum_{j=1}^{d} \int_{B_\delta(x_j)} e_\varepsilon(\widetilde{U}_\varepsilon)\,dx \\ &\leqq \int_{A_\varepsilon} e_\varepsilon(\widetilde{U}_\varepsilon)dx + C(\lambda, K) \end{aligned} \tag{4.3}$$

where

$$A_\varepsilon = \bigcup_{j=1}^{d} B_\delta(x_j) \Big\backslash \bigcup_{j \geqq d+1} B_j .$$

Here the first inequality is true because

$$\int_{B_j} e_\varepsilon(\widetilde{U}_\varepsilon)dx = \int_{B_j} e_\varepsilon(u_\varepsilon)\,dx \geqq \pi d \log \frac{1}{\varepsilon}(1-\alpha_j) - C(\lambda, K)$$

and

$$\int_{B_\delta(x_j)\setminus B_j} e_\varepsilon(\widetilde{U}_\varepsilon)\,dx \geqq \frac{1}{2}\int_{B_\delta(x_i)|B_j} |\nabla \widetilde{U}_\varepsilon|^2\,dx$$
$$\geqq \pi d \log \frac{\delta}{\varepsilon^{\alpha_j}} - C(\lambda, K) \quad \text{for } j = 1, 2, \ldots, d;$$

see [26], where, in the second inequality of (4.3), we have used Lemma 2.1 for $\widetilde{U}_\varepsilon$ on each B_j, $d+1 \leqq j \leqq N_\varepsilon \leqq N_*(\lambda, K)$.

Since $u_\varepsilon \in S_g(\lambda, K)$, we deduce from (4.3) that

$$\int_{\Omega_\delta \cup \left(\bigcup_{j=d+1}^{N_\varepsilon} B_j\right)} e_\varepsilon(u_\varepsilon)\,dx \leqq C(\lambda, K) + \pi d \log \frac{1}{\delta} \tag{4.4}$$

where $\Omega_\delta = \Omega \setminus \bigcup_{j=1}^{d} B_\delta(x_j)$.

Now, for a sequence of $\varepsilon_n \downarrow 0$, we may assume, without loss of generality, that $x_j \to b_j$ as $\varepsilon_n \downarrow 0$. Note that x_j may also depend on ε. We may also assume, by taking a subsequence if necessary, that $u_{\varepsilon_n}(x) \to u^*(x)$ weakly in $H'_{\text{loc}}(\overline{\Omega}\setminus\{b_1, \ldots, b_d\})$ and strongly in $L^2(\Omega)$. The conclusion that

$$u^*(x) = \prod_{j=1}^{d} \frac{x - b_j}{|x - b_j|} e^{ih(x)} \quad \text{with } h(x) \in H'(\Omega)$$

follows from the proof of Theorem 3.3.

Remark 4.2. The claim made in the proof of Theorem 3.7 follows from Lemma 4.1.

Remark 4.3. Let $\Omega_\varepsilon = \Omega \setminus \cup_{j=1}^{N_\varepsilon} B_j$; then

$$\sum_{j=d+1}^{N_\varepsilon} \int_{B_j} e_\varepsilon(U_\varepsilon)\,dx + \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} (1 - |u_\varepsilon|^2)^2\,dx \leqq C(\lambda, K). \tag{4.5}$$

In fact, if we choose a suitable $\delta \geqq \frac{1}{2}\delta(c, K)$ in the proof of Lemma 4.1, then one has

$$\int_{\Omega \setminus \bigcup_{j=1}^{d} B_j} e_\varepsilon(u_\varepsilon)dx \leqq \pi \log \frac{1}{\varepsilon} \sum_{j=1}^{d} \alpha_j + C(\lambda, K)$$

and

$$\sum_{j=1}^{d} \pi \log \frac{\delta}{\varepsilon^{\alpha_j}} - C(\lambda, K) \leqq \sum_{j=1}^{d} \int_{B_\delta(x_j) \setminus B_j} \frac{1}{2} |\nabla \widetilde{U}_\varepsilon|^2$$

$$\leqq \int_{\bigcup_{j=1}^{d} B_\delta(x_j) \setminus \bigcup_{j=1}^{N_\varepsilon} B_j} \frac{1}{2} |\nabla \widetilde{U}_\varepsilon|^2 + C(\lambda, K)$$

(again using Lemma 2.1)

$$\leqq \int_{\Omega_\varepsilon} \frac{1}{2} |\nabla u_\varepsilon|^2 \, dx \leqq \int_{\Omega \setminus \bigcup_{j=1}^{d} B_j} e_\varepsilon(u_\varepsilon)$$

$$\leqq \sum_{j=1}^{d} \alpha_j \pi \log \frac{1}{\varepsilon} + C(\lambda, K).$$

The estimate (4.5) then follows.

Let us now consider (4.2). Taking an arbitrary sequence $\{\varepsilon_n\}, \varepsilon_n \downarrow 0$, we may find a subsequence of $\{u_{\varepsilon_n}(x,t)\}$, still denoted by $\{u_{\varepsilon_n}\}$, such that

(i) $$\left\| \Delta u_{\varepsilon_n} + \frac{1}{\varepsilon_n^2} u_{\varepsilon_n} \left(1 - |u_{\varepsilon_n}|^2\right) \right\|_{L^2(\Omega)} (t) \to 0 \text{ a.e. } t,$$

as a function of t when $\varepsilon_n \to 0$, and

(ii) $$\|\nabla u_{\varepsilon_n}\|(t) \left\| \Delta u_{\varepsilon_n} + \frac{1}{\varepsilon_n^2} u_{\varepsilon_n} \left(1 - |u_{\varepsilon_n}|^2\right) \right\| (t) \to 0 \text{ a.e. } t \text{ as } \varepsilon_n \to 0$$

Here we used the fact that $\|\nabla u_{\varepsilon_n}(t)\|^2 \leqq C \log \frac{1}{\varepsilon_n}$,

$$\int_0^\infty \left\| \Delta u_{\varepsilon_n} + \frac{1}{\varepsilon_n^2} u_{\varepsilon_n} \left(1 - |u_{\varepsilon_n}|^2\right) \right\|^2 (t) dt \leqq C(K)/T_n,$$

and $T_{\varepsilon_n} / \log \frac{1}{\varepsilon_n} \to \infty$ as $\varepsilon_n \to 0$.

4.1. Class $S(t)$

Definition 4.4. For the given sequence $\{u_{\varepsilon_n}(x,t)\}$ and for any t, we introduce a function class $S(t)$ that, in principle, may also depend on the choices of $\{\varepsilon_n\}$.

We say a function $V(x)$ of form $\prod_{j=1}^{d} \frac{x-a_j}{|x-a_j|} e^{ih_a(x)}$ belongs to $S(t)$ if there is a subsequence of $\{u_n(x)\}$, $u_n(x) = u_{\varepsilon_n}(x,t)$, that converges to $V(x)$ in $L^2(\Omega)$ and weakly in $H^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{a_1, \dots, a_d\})$.

By Theorem 2.4, the proof of Theorem 3.3, and Lemma 4.1, we conclude that

$$\min\{|a_i - a_j|, \operatorname{dist}(a_i, \partial\Omega) : i \neq j, \ i,j = 1, \dots, d\} \geqq \delta(\lambda, K) > 0$$

and that

$$\|h_a(x)\|_{H'(\Omega)} \leqq C(\lambda, K)$$

for all $V \in S(t)$.

Moreover, for a.e. t, from Theorem 3.7 we have $\Delta h_a(x) = 0$ for all $V \in S(t)$. We want to show next that, for a.e. t and all $V \in S(t)$, the corresponding $a = (a_1, \ldots, a_d)$ is a critical point of $W_g(\cdot)$ in Ω^d. To do so, we let $\{u_n\}$ be the sequence of $\{u_{\varepsilon_n}(x,t)\}$ with t satisfying (ii) and $\{u_n\}$ converging to $V \in S(t)$.

Multiplying $\Delta u_n + \frac{1}{\varepsilon_n^2} u_n(1 - |u_n|^2)$ by $\frac{\partial u_n}{\partial x_k}$ and integrating over $B_R(a_j)$ for some small, suitable R and a given a_j, we find

$$\begin{aligned} -\int_{\partial B_R} \frac{\partial u_n}{\partial \nu}\frac{\partial u_n}{\partial x_k} + \frac{1}{2}\int_{\partial B_R} |\nabla u_n|^2 (\nu \cdot e_k) \\ = -\frac{1}{4\varepsilon_n^2}\int_{\partial B_R} \left(|u_n|^2 - 1\right)^2 (\nu \cdot e_k) + o(1), \end{aligned} \tag{4.6}$$

where $o(1)$ is a quantity that goes to zero as $n \to \infty$. This follows from the fact that (ii) is valid for t.

As $n \to \infty$ and for a suitable $R \in (0, \delta(\lambda, K))$, we claim the following is true:

$$u_n \to V \text{ strongly in } H'(\partial B_R) \tag{4.7a}$$

and

$$\frac{1}{\varepsilon_n^2}\int_{\partial B_R} \left(|u_n|^2 - 1\right)^2 \to 0 \tag{4.7b}$$

for a subsequence of $n \to \infty$.

Let us assume (4.7) for the moment and proceed with our proof that a is a critical point of $W_g(\cdot)$. Let $e^{i\theta(x)} = \frac{x-a_j}{|x-a_j|}$, and write $V(x)$ as $e^{i(\theta(x)+H(x))}$; then $\Delta H(x) = 0$ in $B_{\delta(c,K)}(a_j)$. Because of (4.6) and (4.7), we have

$$0 = -\int_{\partial B_R} \frac{\partial V}{\partial \nu} \cdot \frac{\partial V}{\partial x_k} + \frac{1}{2}\int_{\partial B_R} |\nabla V|^2 (\nu \cdot e_k). \tag{4.8}$$

A direct calculation (see pp. 74–75 in [2]) implies

$$0 = -\int_{\partial B_R} \frac{\partial H}{\partial \nu}\left(\frac{\partial H}{\partial x_k} + \frac{\tau \cdot e_k}{R}\right) + \int_{\partial B_R} \left(\frac{1}{R}\frac{\partial H}{\partial \tau} + \frac{1}{2}|\nabla H|^2\right) \nu \cdot e_K .$$

This and the fact that $\Delta H = 0$ in B_R implies $\nabla H(0) = 0$. Therefore, $\nabla W_g(a) = 0$ by Theorem VIII.3 in [2].

4.2. Proof of (4.7)

Since $u_n \in S_g(\lambda, K)$ and $u_n(x)$ converges to

$$V(x) = \prod_{j=1}^{d} \frac{x - a_j}{|x - a_j|} e^{ih_a(x)},$$

we deduce from Theorem 2.4 and the proof of Lemma 4.1 (see (4.4) and (4.5)) that there is an $R \in [\delta/2, \delta], \delta = \delta(\lambda, K)$ such that the annular domains $A_{(j)}$,

$$A_{(j)} = \left\{ x : R - \frac{\delta}{4N_*} \leqq |x - a_j| \leqq R + \frac{\delta}{4N_*} \right\}, \quad j = 1, 2, \ldots, d,$$

has the property that

$$\sum_{j=1}^{d} \int_{A_{(j)}} e_\varepsilon(u_n) dx \leqq C(\lambda, K). \tag{4.9}$$

Moreover, for all sufficiently large n (again by taking a subsequence of such n's if necessary), $|u_n(x)| \geqq \frac{1}{2}$ on $\bigcup_{j=1}^{d} A_{(j)}$. The latter follows from the fact that those balls B_j, $j = 1, \ldots, N_\varepsilon$, in Theorem 2.4 for u_n converge to a set consisting of at most N_* points.

Next, for each point y, $|y - a_j| = R, j = 1, \ldots, d$, we let B be a ball centered at y and of radius

$$\rho \in \left[\frac{\delta}{8N_*}, \frac{\delta}{4N_*} \right]$$

such that

$$\int_B e_\varepsilon(u_n) dx + \int_{\partial B} e_\varepsilon(u_n) \leqq C(\lambda, K), \tag{4.10}$$

is valid for a subsequence of $n \to \infty$ (see p. 57 in [23]).

From (4.10), it is not hard to derive that $\deg(u_n, \partial B) = 0$ and that $\| |u_n| - 1 \|(x) \leqq C(\lambda, K) \varepsilon_n^{\frac{1}{4}}$ for $x \in \partial B$. We want to show next that the subsequences of u_n that satisfy (4.10) and

$$\left\| \Delta u_n + \frac{1}{\varepsilon_n^2} u_n \left(1 - |u_n|^2\right) \right\|_{L^2(\Omega)} \to 0$$

are strongly convergent in $H'(B)$, and

$$\frac{1}{\varepsilon_n^2} \int_B \left(|u_n|^2 - 1 \right) dx \to 0,$$

as $n \to +\infty$.

For this purpose, we write $u_n = \rho_n e^{i\psi_n}$ on B so that ψ_n and ρ_n satisfy the following equations:

$$\operatorname{div}\left(\rho_n^2 \nabla \psi_n\right) = f_n \in L^2(B), \quad \psi_n|_{\partial B} \in H'(\partial B), \tag{4.11}$$

and

$$\Delta \rho_n + \frac{\rho_n}{\varepsilon_n^2}(1 - \rho_n^2) - |\nabla \psi_n|^2 \rho_n = g_n \in L^2(B) \tag{4.12}$$

with

$$|\rho_n - 1|(x) \leqq C(\lambda, K)\varepsilon_n^{\frac{1}{4}}, \quad x \in \partial B, \quad \int_{\partial B} |\nabla \rho_n|^2 + \frac{1}{\varepsilon_n^2}(1-\rho_n^2)^2 \leqq C(\lambda, K),$$

and

$$|\rho_n(x)| \geqq \frac{1}{2} \quad \text{on } B$$

where $\|f_n\|_{L^2(B)} + \|g_n\|_{L^2(B)} \to 0$ as $n \to \infty$ (cf. (ii)).

It is then easy to see $\psi_n \overset{H'(B)}{\rightharpoonup} \psi_0$ as $n \to \infty$ where $\Delta\psi_0 = 0$ in B, and $\psi_0|_{\partial B}$ is the weak limit of $\psi_n|_{\partial B}$ in $H'(\partial B)$. By the reverse Hölder inequality and a Caccioppoli-type estimate for $|\nabla\psi_n|^2$, we see $g_n + \rho_n|\nabla\psi_n|^2 \in L^p(B)$ for a fixed $p > 1$. This combines with (4.12) and the boundary estimate $1 - \rho_n \leqq C(\lambda, K)\varepsilon_n^{\frac{1}{4}}$ to imply that $1 - \rho_n \leqq C\varepsilon_n^{\beta}$ for some constants β and C.

We then multiply equation (4.12) by ρ_n and integrate it over B. Then

$$\begin{aligned} \frac{1}{4\varepsilon_n^2}\int_B (1-\rho_n^2)dx &\leqq \int_B |\nabla\rho_n|^2 + |\nabla\psi_n|^2 + |\rho_n g_n| \\ &\quad + \int_{\partial B} \frac{\partial \rho_n}{\partial \nu}\cdot \rho_n \leqq C(\lambda, K). \end{aligned} \tag{4.13}$$

Therefore,

$$\frac{1}{\varepsilon_n^2}\int_B (1-\rho_n^2)^2 dx \leqq C\varepsilon_n^{\beta} \to 0 \quad \text{as } n \to \infty.$$

Next, let $V_n = \rho_n^2 - 1$. Then

$$\begin{cases} -\Delta V_n + \frac{2\rho_n^2}{\varepsilon_n^2}V_n = -2\rho_n^2|\nabla\psi_n|^2 + 2g_n\rho_n - 2|\nabla\rho_n|^2 & \text{in } B, \\ |V_n| \leqq C(\lambda, K)\varepsilon_n^{\frac{1}{4}} & \text{on } \partial B. \end{cases} \tag{4.14}$$

We multiply (4.14) by V_n and integrate it over B to obtain

$$\begin{aligned} &\int_B |\nabla V_n|^2 + \frac{1}{2\varepsilon_n^2}V_n^2 \\ &\leqq -2\int_B V_n\rho_n^2\left(|\nabla\psi_n|^2 + 2|g_n| - 2|\nabla\rho_n|^2\right)dx + \int_{\partial B} V_n\frac{\partial V_n}{\partial \nu} \\ &= o(1) + C(\lambda, K)\int_{\partial B} V_n^2 \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.15}$$

Note that $|\nabla V_n|^2 = |2\rho_n\nabla\rho_n|^2 \geqq |\nabla\rho_n|^2$; we conclude that $u_n \to V$ in $H'(B)$. We cover the annular domains

$$\left\{x : R - \frac{\delta}{16N_*} \leqq |x - a_j| \leqq R + \frac{\delta}{16N_*}\right\}, \quad j = 1, \ldots, d,$$

by a finite number of such B's as above and find a subsequence of u_n such that it strongly converges to V on these annular domains and

$$\frac{1}{\varepsilon_n^2}\sum_{j=1}^{d}\int_{R-\frac{\delta}{16N_*}\leqq|x-a_j|\leqq R+\frac{\delta}{16N_*}}\left(|u_n|^2-1\right)^2dx$$

goes to zero as $n\to\infty$.

Finally, we apply Fubini's theorem to obtain (4.7).

4.3. Summary of Findings

To summarize, we have proved the following:

THEOREM 4.5. *Let $u_\varepsilon(x,t)$ be the solution of* (4.2) *with the initial data u_0^ε satisfying Assumption* 3.1. *Then, for any sequence of $\varepsilon_n\downarrow 0$, there is a subsequence of $\{\varepsilon_n\}$, still denoted by $\{\varepsilon_n\}$, such that the corresponding sequence $\{u_{\varepsilon_n}(x,t)\}$ has the properties*

(i) *for any $t>0$,* $\operatorname{dist}(u_{\varepsilon_n}(x,t),S(t))\to 0$ *as $n\to\infty$; and*
(ii) *for almost all $t>0$ and all $V\in S(t)$,*

$$V(x)=\prod_{j=1}^{d}\frac{x-a_j}{|x-a_j|}e^{ih_a(x)},\quad \Delta h_a(x)=0,$$

and $a=(a_1,\dots,a_d)$ is a critical point of $W_g(\cdot)$ where $\operatorname{dist}(U,S(t))=\inf\{\|u-V\|_{L^2(\Omega)}:V\in S(t)\}$.

Remark 4.6. $\{S(t):t>0\}$ is dependent on the starting choices of $\varepsilon_n\downarrow 0$. We do not know if part (ii) of Theorem 4.5 remains true if we define $S(t)$ to be the set of all possible ω-limits of $\{u_\varepsilon(\cdot,t),\varepsilon>0\}$, that is, limits of any sequence $\{u_{\eta_n}(\cdot,t)\}$, $\eta_n\downarrow 0$.

COROLLARY 4.7. *Assume the hypothesis of Theorem* 4.5 *and suppose that $W_g(\cdot)$ has a unique critical point (which has to be the global minimum point of $W_g(\cdot)$). Then, as $\varepsilon\to 0$,*

$$u_\varepsilon(x,t)\to\prod_{j=1}^{d}\frac{x-a_j}{|x-a_j|}e^{ih(x)}$$

in $L^2_{\mathrm{loc}}(\bar\Omega\times\mathbb{R}_+)$ where $a=(a_1,\dots,a_d)$ is the critical point of $W_g(\cdot)$ and $\Delta h(x)=0$ in Ω with $h(x)=h_0(x)$ on $\partial\Omega$.

Proof: It follows from Theorem 4.5 and the same arguments as in the proof of Theorem 3.7.

Remark 4.8. Theorem 4.5 and Corollary 4.7 show that the mobilities of vortices cannot be much larger than a multiple of $\log\frac{1}{\varepsilon}$. In [17], we also showed that when the initial data u_0^ε has vortices located near a local nondegenerate minimum point $a = (a_1, \dots, a_d)$ of $W_g(\cdot)$ and the energy $E_\varepsilon(u_\varepsilon^0)$ is close to $\pi d \log\frac{1}{\varepsilon} + \gamma d + W_g(a)$ (cf. Corollary 2.6), then the vortices of $u^\varepsilon(x,t)$, $t > 0$, the solution of (1.1)–(1.3), will stay near the given vortices $a_1, \dots, a_d$ for all time $t > 0$.

5. The $\log\frac{1}{\varepsilon}$–Scale Time Dynamics

In Section 3 we showed that the vortices of $u_\varepsilon(\cdot,t), t > 0$, do not move much in the initial time interval $0 \leqq t \leqq \delta \log\frac{1}{\varepsilon}$ whenever ε is sufficiently small and δ is a small number where $u_\varepsilon(x,t)$ are solutions of (1.1)–(1.3) with initial data satisfying Assumptions 3.1 and 3.2. We also showed, in Section 4, that all the dynamics are essentially finished after a time of size $M \log\frac{1}{\varepsilon}$, where M is a sufficiently large number. Since the initial vortices, say $b_1, b_2, \dots, b_d$, may not be critical points of $W_g(\cdot)$, and since after a time much larger than $\log\frac{1}{\varepsilon}$ all the vortices form a critical point of $W_g(\cdot)$, we conclude that vortices have to move in the $\log\frac{1}{\varepsilon}$ time scale. Thus we consider the evolution equations:

$$
(5.1) \qquad \begin{cases} \frac{1}{\log\frac{1}{\varepsilon}}\frac{\partial}{\partial t} u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon \left(1 - |u_\varepsilon|^2\right) & \text{in } \Omega \times \mathbb{R}_+\,, \\ u_\varepsilon(x,t) = g(x) & \text{on } \partial\Omega \times \mathbb{R}_+\,, \\ u_\varepsilon(x,0) = u_0^\varepsilon(x)\,. \end{cases}
$$

As we mentioned in the introduction, various formal asymptotic matching arguments (cf. [7] and [18]) show that the vortices $a(t) = (a_1(t), \dots, a_d(t))$ of a solution $u_\varepsilon(\cdot,t)$ of (5.1) satisfy

$$
(5.2) \qquad \frac{d}{dt} a(t) = -\operatorname{grad} W_g(a)\,.
$$

We are still not able to prove (5.2) is the case, even though our analysis indicates that such a law should be true. (This is shown in the remark following this paper.) We do, however, have the following theorem.

THEOREM 5.1. *Let $u_\varepsilon(x,t)$, $\varepsilon > 0$, be solutions of* (5.1) *with the initial data u_0^ε satisfying Assumptions* 3.1 *and* 3.2, *and let $\{u_{\varepsilon_n}\}$, $\varepsilon_n \downarrow 0$, be a sequence of such solutions. Suppose that $S(t)$ is the class associated with the sequence $\{u_{\varepsilon_n}\}$ (see the definition in Section* 4*) for $t \geqq 0$. We define*

$$
(5.3) \qquad A(t) = \left\{ a(t) = (a_1(t), \dots, a_d(t)) : \text{There is a } V \in S(t) \text{ of the form } V(x) = \prod_{j=1}^{d} \frac{x - a_j(t)}{|x - a_j(t)|} e^{ih(x)} \right\}.
$$

Then we have Hausdorff distance $(A(t),\ A(t+\Delta t)) \leqq \eta(\Delta t)$ *for all* $t > 0,\ \Delta t > 0$, *and* $\eta(\Delta t) \to 0$ *as* $\Delta t \to 0$; *and*

$$\tag{5.4} \begin{aligned} &\int_t^{t+\Delta t} \inf_{a(\tau)\in A(\tau)} |\operatorname{grad} W_g(a(\tau))|^2\, d\tau \\ &\leqq \pi \cdot \varliminf_{\varepsilon_n \to 0} \left[E_{\varepsilon_n}\left(u_{\varepsilon_n}(\cdot,t)\right) - E_{\varepsilon_n}\left(u_{\varepsilon_n}(\cdot,t+\Delta t)\right)\right]. \end{aligned}$$

Moreover, we may choose a suitable subsequence of $\{u_{\varepsilon_n}\}$, *call it* $\{V_n\}$, *such that the class* $S'(t)$ *associated with* $\{V_n\}$ *has the property that, for each* $V \in S'(t)$,

$$V(x) = \prod_{j=1}^{d} \frac{x-a_j}{|x-a_j|}\, e^{ih_a(x)},$$

for some d *distinct points* $a_1, \dots, a_d$ *in* Ω. *That is, the corresponding* $A'(t)$ *consists of one point.*

Remark 5.2. The function $\inf_{a(\tau)\in A(\tau)} |\operatorname{grad} W_g(a(\tau))|^2$ inside the integral of (5.4) is a continuous function of $\tau \in [0,\infty)$. In fact, it simply follows from the continuity in the Hausdorff metric of these $A'(t)$'s and the continuity of grad $W_g(\cdot)$ near all $A(\tau)$, $\tau \in [0,\infty)$.

We also note that

$$\tag{5.5} \begin{aligned} &\varliminf_{\varepsilon \to 0^+} \left[E_\varepsilon(u_\varepsilon(\cdot,t)) - E_\varepsilon(u_\varepsilon(\cdot,t+\Delta t))\right] \\ &= \lim_{\varepsilon_n \to 0} \left[E_{\varepsilon_n}\left(u_{\varepsilon_n}(\cdot,t)\right) - E_{\varepsilon_n}\left(u_{\varepsilon_n}(\cdot,t+\Delta t)\right)\right] \end{aligned}$$

for a suitable sequence of $\varepsilon_n \downarrow 0$ (which may depend on t and $t+\Delta t$). If we let $S^*(t)$ be the class associated with the family $\{u_\varepsilon,\ \varepsilon > 0\}$, that is, all w-limits of $\{u_\varepsilon,\ \varepsilon > 0\}$, and let $A^*(t)$ be the corresponding set for each $t > 0$, then, by our choice of $\{u_{\varepsilon_n}\}$,

$$\tag{5.6} \begin{aligned} &\int_t^{t+\Delta t} \inf_{a(\tau)\in A^*(\tau)} |\operatorname{grad} W_g(a(\tau))|^2\, d\tau \\ &\leqq \int_t^{t+\Delta t} \inf_{a(\tau)\in A(\tau)} |\operatorname{grad} W_g(a(\tau))|^2\, d\tau \\ &\leqq \pi \cdot \varliminf_{\varepsilon_n \to 0} \left[E_{\varepsilon_n}\left(u_{\varepsilon_n}(\cdot,t)\right) - E_{\varepsilon_n}\left(u_{\varepsilon_n}(\cdot,t+\Delta t)\right)\right] \\ &= \pi \lim_{\varepsilon_n \to 0} \left[E_{\varepsilon_n}\left(u_{\varepsilon_n}(\cdot,t)\right) - E_{\varepsilon_n}\left(U_{\varepsilon_n}(\cdot,t+\Delta t)\right)\right] \\ &= \pi \varliminf_{\varepsilon \to 0} \left[E_\varepsilon(u_\varepsilon(\cdot,t)) - E_\varepsilon(u_\varepsilon(\cdot,t+\Delta t))\right]. \end{aligned}$$

Proof of Theorem 5.1: The final statement of Theorem 5.1 follows from the same line of argument used in proving Theorem 3.7. It therefore suffices to verify both (5.4) and the continuity in the Hausdorff metric of sets $A(t)$ in t.

To show the continuity of sets $A(t)$ in the Hausdorff metric, we let $a = (a_1, \dots, a_d) \in A(t)$. Thus there is a subsequence $\{V_n\}$ of $\{u_{\varepsilon_n}\}$, $v_n = u_{\varepsilon'_n}$, such that

$$V_n(x) \to v(x) = \prod_{j=1}^{d} \frac{x - a_j}{|x - a_j|} e^{ih(x)} \quad \text{in } L^2(\Omega)$$

and weakly in $H^1_{\text{loc}}(\overline{\Omega} | \{a_1, \dots, a_d\})$ (cf. Lemma 4.1). Then we consider (5.1) with the initial data $u_0^\varepsilon(x) = u_\varepsilon(x,t)$, $\varepsilon = \varepsilon'_n$. Let

$$\phi(x) \in C^1\ (\overline{\Omega}) \quad \text{with } \phi(x) \equiv 0 \text{ on } \bigcup_{j=1}^{d} B_\delta(a_j),$$

and

$$\phi(x) \equiv 1 \quad \text{on } \Omega \Big| \bigcup_{j=1}^{d} B_{2\delta}(a_j),\ \delta > 0$$

and $\delta < \delta(\lambda, K)$ (see Theorem 2.4).

By using calculations similar to those in (3.4) through (3.7), we obtain

$$\begin{aligned} &\int_\Omega \phi^2(x) e_\varepsilon\,(u_\varepsilon(\cdot, t + \Delta t))\, dx \\ &\qquad \leqq C(K)\Delta t \log \frac{1}{\varepsilon} + C(K), \quad \varepsilon = \varepsilon'_n. \end{aligned} \tag{5.7}$$

(cf. also (3.14) through (3.16) with $\lambda_\varepsilon = \log \frac{1}{\varepsilon}$). When Δt is a sufficiently small (independent of ε'_n) positive number, any w-limit of the sequence $\{u_{\varepsilon'_n}(\cdot, t + \Delta t)\}$ is a function $\bar{V}$ of the form

$$\bar{V}(x) = \prod_{j=1}^{d} \frac{x - \bar{a}_j}{|x - \bar{a}_j|} e^{i\bar{h}}$$

(cf. Lemma 4.1). Moreover, $|a - \bar{a}| \leqq \eta(\Delta t) \to 0$ as $\Delta t \to 0$. Here $\bar{a} = (\bar{a}_1, \dots, \bar{a}_d)$, and the estimate $|a - \bar{a}| \leqq \eta(\Delta t)$ is a consequence of Lemma 2.1, Theorem 2.4, and the estimate (5.7). In particular, there is an $\bar{a} \in A(t + \Delta t)$ such that $|a - \bar{a}| \leqq \eta(\Delta t)$. This shows that $A(t)$ is contained in an $\eta(\Delta t)$-neighborhood of $A(t + \Delta t)$.

Conversely, let $\bar{a} \in A(t + \Delta t)$, and let

$$u_n(x) = u_{\varepsilon'_n}(x, t + \Delta t) \to \bar{V}(x) = \prod_{j=1}^{d} \frac{x - \bar{a}_j}{|x - \bar{a}_j|} e^{ih};$$

now consider the subsequence $\{u_{\varepsilon'_n}(\cdot, t)\}$ of $\{u_{\varepsilon_n}(\cdot, t)\}$. By Lemma 4.1 again, we may find a subsequence of $\{u_{\varepsilon'_n}(\cdot, t)\}$, which we shall still denote it by $\{V_n\}$, that converges to a map $v(x)$ of the form

$$\prod_{j=1}^{d} \frac{x - a_j}{|x - a_j|} e^{ih(x)}$$

We consider again (5.1) with the initial data V_n (and corresponding ε's). Then, from the above proof, we see $|a - \bar{a}| \leqq \eta(\Delta t)$ is valid. Thus $A(t + \Delta t)$ is also contained in an $\eta(\Delta t)$-neighborhood of $A(t)$. This completes the proof of (5.3).

To show (5.4), we employ some of the arguments used in the proof of Theorem 4.5. In particular, the proof of the statements (4.7a) and (4.7b) will be needed. We want to show

$$\begin{aligned}(5.8)\qquad \inf_{a(\tau)\in A(\tau)} |\operatorname{grad} W_g(a(\tau))|^2 &\leqq \pi \cdot \varliminf_{\varepsilon_n\to 0} \int_\Omega \left|\frac{\partial u_{\varepsilon_n}}{\partial t}\right|^2 dx \Big/ \log\frac{1}{\varepsilon_n} \\ &= \pi \varliminf_{\varepsilon_n\to 0}\left[-\frac{d}{dt}E_{\varepsilon_n}\left(u_{\varepsilon_n}(\cdot,t)\right)\right] \quad \text{for all } \tau\in[0,\infty).\end{aligned}$$

If (5.8) is proved, then (5.4) follows simply by an integration and Fatou's lemma from real variable theory.

To show (5.8) for all $\tau \in [0,\infty)$, we may assume that the right-hand side of (5.7) is not infinite, for otherwise the inequality is trivially valid. That is, we may assume

$$\begin{aligned}(5.9)\qquad &\varliminf_{\varepsilon_n\to 0}\int_\Omega \left|\frac{\partial u_{\varepsilon_n}}{\partial t}\right|^2(\tau)\,dx \Big/ \log\frac{1}{\varepsilon_n} \\ &= \varliminf_{\varepsilon_n\to 0}\int_\Omega \left|\Delta u_{\varepsilon_n} + \frac{u_{\varepsilon_n}}{\varepsilon_n^2}\left(1 - |u_{\varepsilon_n}|^2\right)\right|^2(\tau)\log\frac{1}{\varepsilon_n}\,dx \\ &= \rho(\tau) < \infty.\end{aligned}$$

Let $u_n(x) = u_{\varepsilon_n'}(x,\tau)$, $n = 1,2,\dots$, be a subsequence of $\{u_{\varepsilon_n}(\cdot,t)\}$ that satisfies (5.9). By Lemma 4.1, we may also assume (by taking a subsequence of $\{u_n\}$ if necessary) that

$$u_n(x) \to v(x) = \prod_{j=1}^{d}\frac{x - a_j}{|x - a_j|}e^{ih_a(x)} \quad \text{in } L^2(\Omega)$$

and weakly in

$$H^1_{\mathrm{loc}}\left(\overline{\Omega}\backslash\{a_1,\dots,a_d\}\right).$$

As a result of (5.9), we have $\Delta h_a(x) = 0$ in Ω. As in the proof of Theorem 4.5, we choose d balls, $B_R(a_j)$, $j = 1,\dots,d$, for some suitable $R > 0$. Then we multiply $\Delta u_\varepsilon + \frac{1}{\varepsilon^2}u_\varepsilon(1 - |u_\varepsilon|^2)$ by $\frac{\partial u_\varepsilon}{\partial x_k}$ and then integrate it over $B_R(a_j)$ for $j = 1,2,\dots,d$. Here $\varepsilon = \varepsilon_n'$. We obtain

$$\begin{aligned}(5.10)\qquad &-\int_{\partial B_R(a_j)}\frac{\partial u_\varepsilon}{\partial \nu}\cdot\frac{\partial u_\varepsilon}{\partial x_k} + \frac{1}{2}\int_{\partial B_R(a_j)}\left|\nabla u_\varepsilon\right|^2 \nu\cdot e_k \\ &= -\frac{1}{4\varepsilon^2}\int_{\partial B_R(a_j)}\left(|u_\varepsilon|^2 - 1\right)^2 \nu\cdot e_k + \frac{1}{\log\frac{1}{\varepsilon}}\int_{B_R(a_j)}\frac{\partial u_\varepsilon}{\partial t}\cdot\frac{\partial u_\varepsilon}{\partial x_k}\,dx\end{aligned}$$

(cf. Theorem 4.5), $\varepsilon = \varepsilon_n'$, $n = 1, 2, \ldots$.

We note that statements (4.7a) and (4.7b) are both true in the present situation if we substitute $u_{\varepsilon_n'}(x, \tau)$ for u_n. Indeed, in all estimates (4.9) through (4.12) as well as in estimates (4.13) through (4.15), we need only the assumption that

$$\left\| \Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \right\|_{L^2(\Omega)} (\tau), \quad \varepsilon = \varepsilon_n', \ n = 1, 2, \ldots,$$

are uniformly bounded. From (5.9), this assumption is certainly justified.

On the other hand, for $k = 1, 2$, we have

$$\text{(5.11)} \quad \begin{aligned} &\int_{B_R(a_j)} \left| \frac{\partial u_\varepsilon}{\partial t} \right| \left| \frac{\partial u_\varepsilon}{\partial t_k} \right| dx \Big/ \log \frac{1}{\varepsilon} \\ &\leqq \left[\int_{B_R(a_j)} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 \Big/ \log \frac{1}{\varepsilon}\, dx \right]^{1/2} \left[\int_{B_R} \left| \frac{\partial u_\varepsilon}{\partial x_k} \right|^2 \Big/ \log \frac{1}{\varepsilon} \right]^{1/2}. \end{aligned}$$

As $n \to \infty$ and for $k = 1, 2$, the left-hand side of the identity (5.10) converges to

$$\text{(5.12)} \quad -\int_{\partial B_R(a_j)} \frac{\partial v}{\partial \nu} \cdot \frac{\partial v}{\partial x_k} + \frac{1}{2} \int_{\partial B_R(a_j)} |\nabla v|^2 \nu \cdot e_k$$

as in (4.8). By calculations in the proof of Theorem VIII.3 of [2], (5.12) is exactly the gradient of $W_g(\cdot)$ with respect to the a_j-variable at the point $a = (a_1, \ldots, a_d)$, which is equal to 2π grad $H_j(a_j)$. Here we write $v(x) = e^{i\theta_j(x)}\, e^{iH_j(x)}$ for x near a_j and $e^{i\theta_j(x)} = \frac{x - a_j}{|x - a_j|}$. Therefore, we conclude

$$\text{(5.13)} \quad \begin{aligned} |\operatorname{grad} W_g(a)|^2 &\leqq \lim_{\varepsilon = \varepsilon_n' \to 0} \sum_{j=1}^{d} \left[\int_{B_R(a_j)} \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 (\tau)\, dx \Big/ \log \frac{1}{\varepsilon} \right. \\ &\qquad \left. \times \int_{B_R(a_j)} |\nabla u_e|^2 (\tau)\, dx \Big/ \log \frac{1}{\varepsilon} \right] \\ &\leqq \pi \lim_{\varepsilon = \varepsilon_n' \to 0} \int_\Omega \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 (\tau)\, dx \Big/ \log \frac{1}{\varepsilon}. \end{aligned}$$

In the last inequality, we have used the fact that

$$\int_{B_R(a_j)} |\nabla u_\varepsilon|^2 (\tau)\, dx \Big/ \log \frac{1}{\varepsilon} = \pi + o(1)$$

(cf. the proof of Lemma 4.1). Now the inequality (5.8) follows from (5.13). We have thus completed the proof of (5.4).

We can now state two consequences of the proof for Theorem 5.1 and statement (5.4) and the continuity of the $A(t)$'s in the Hausdorff metric.

COROLLARY 5.3. *Let $\{u_n\}$ be a sequence of maps in the class $S_g(\lambda, K)$ with corresponding $\varepsilon = \varepsilon_n$, $n = 1, 2, \ldots$. Suppose that*

$$u_n(x) \to v(x) = \prod_{j=1}^{d} \frac{x - a_j}{|x - a_j|} e^{ih_a(x)} \quad \text{in } L^2(\Omega).$$

Then

$$|\operatorname{grad} W_g(a)|^2 \leqq \pi \cdot \lim_{\varepsilon_n \to 0} \log \frac{1}{\varepsilon_n} \|M_{\varepsilon_n}\|^2_{L^2(\Omega)},$$

where

$$M_{\varepsilon_n} = \Delta u_{\varepsilon_n} + \frac{1}{\varepsilon_n^2} u_{\varepsilon_n}(1 - |u_{\varepsilon_n}|^2).$$

Remark 5.4. The above result gives a proof of the observation made in [9].

Proof: If

$$\lim_{\varepsilon_n} \log \frac{1}{\varepsilon_n} \|M_{\varepsilon_n}\|^2_{L^2(\Omega)} = +\infty,$$

then there is nothing to show. If the infimum is finite, then we follow the last part of the proof of Theorem 5.1 to obtain the conclusion of Corollary 5.3. We should note that in the latter case $\Delta h_a = 0$ in Ω.

COROLLARY 5.5. *Let $u_\varepsilon(x,t)$, $\varepsilon > 0$, be solutions of* (5.1) *with the initial data u_0^ε satisfying Assumptions* 3.1 *and* 3.2, *and let*

$$\delta(\varepsilon) = E_\varepsilon(u_0^\varepsilon) - \lim_{t \to \infty} E_\varepsilon(u_\varepsilon(\cdot, t)), \quad \varepsilon > 0.$$

There is a positive constant δ_0 (depending only on $W_g(\cdot)$ and K) and a positive constant ε_0 depending on K, g, and Ω such that, for all $0 < \varepsilon \leqq \varepsilon_0$, either $\delta(\varepsilon) \geqq \delta_0$ or the following are true:

(i) $|b - a| = \min\{|b - \tilde{a}|, \tilde{a}$ *is a critical point of* $W_g(\cdot)\} \leqq C(\varepsilon, \delta(\varepsilon))$ *for some critical point a of $W_g(\cdot)$. Moreover, $C(\varepsilon, \delta(\varepsilon)) \to 0$ as $\delta(\varepsilon) \to 0$ and $\varepsilon \to 0$.*

(ii) *For any converging sequence $\{u_{\varepsilon_n}(x)\}$, $\varepsilon_n \downarrow 0$, the limit is a function v of the form*

$$v(x) = \prod_{j=1}^{d} \frac{x - \tilde{a}_j}{|x - \tilde{a}_j|} e^{ih_{\tilde{a}}(x)}, \quad \Delta h_{\tilde{a}} = 0 \text{ in } \Omega,$$

where $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_d)$ is a critical point of $W_g(\cdot)$ and $u_{\varepsilon_n}(x) = \lim_{t \to \infty} u_{\varepsilon_n}(x,t)$.

Furthermore, the point $\tilde{a}$ has to lie in the same connected component as a in the critical point set of $W_g(\cdot)$. In particular, $W_g(\tilde{a}) = W_g(a)$.

Proof: Let $F = \{\tilde{a} \in \Omega^d : \tilde{a}$ is a critical point of $W_g(\cdot)$ in $\Omega^d\}$. Since $W_g(\cdot)$ is analytic in a fixed neighborhood of F, say F_{ρ_0}, the ρ_0-neighborhood of F. By the Lojasiewicz inequality (see [24]), we have two positive constants, $\gamma \geqq 2$ and $\sigma_0 \in (0, \rho_0)$, which depend only on $W_g(\cdot)$ such that

$$|\operatorname{grad} W_g(y)| \geqq \min\{\sigma_0, \operatorname{dist}^\gamma(y, F)\}. \tag{5.14}$$

Let us show first that (i) is true when $\delta(\varepsilon) \leqq \delta_0$, $\varepsilon \leqq \varepsilon_0$. Suppose, to the contrary, that (i) is false. Then there would be a sequence of $\varepsilon_n \to 0$, $\delta(\varepsilon_n) \to 0$ and a sequence of points $b_n = (b_1^n, \dots, b_d^n) \in \Omega^d$ such that the initial data $u_{\varepsilon_n}^0$ would satisfy Assumptions 3.1 and 3.2 with $b = (b_1, \dots, b_d) = b_n$ and such that $\operatorname{dist}(b_n, F) \geqq \delta > 0$ for $n = 1, 2, \dots$.

Since the energy bound $E_{\varepsilon_n}(u_{\varepsilon_n}^0) \leqq \pi d \log \frac{1}{\varepsilon_n} + K$, one has $W_g(b_n) \leqq C(K)$. In particular, the $\{b_n\}$ satisfy property (iv) of Theorem 2.4. That is, $b_1^n, \dots, b_d^n$ are d points that lie strictly inside Ω and strictly apart from each other for all $n = 1, 2, \dots$. Thus we may assume, without loss of generality, that $b_n \to b$ as $n \to \infty$. Then $\operatorname{dist}(b, F) \geqq \delta > 0$.

From (5.6), we can deduce that $|\operatorname{grad} W_g(b_n)|^2 \geqq \frac{1}{2} \cdot \min\{\sigma_0^2, \delta^{2\gamma}\}$ for all n sufficiently large.

Let $\Delta t > 0$, which will be chosen later. Then, by (5.6), one has

$$\begin{aligned} &\int_0^{\Delta t} \inf_{a(\tau) \in A^*(\tau)} |\operatorname{grad} W_g(a(\tau))|^2 \, d\tau \\ &\leqq \pi \cdot \varliminf_{n \to \infty} \left[E_{\varepsilon_n}(u_0^{\varepsilon_n}) - E_{\varepsilon_n}(u_{\varepsilon_n}(\cdot, \Delta t))\right] \\ &\leqq \pi \cdot \lim_{n \to \infty} \delta(\varepsilon_n) = 0. \end{aligned} \tag{5.15}$$

On the other hand, (5.3) implies that, for all $0 < \tau \leqq \Delta t$,

$$\sup_{a(\tau) \in A^*(\tau)} |a(\tau) - b| \leqq \eta(\Delta t).$$

Therefore

$$\inf_{a(\tau) \in A^*(\tau)} |\operatorname{grad} W_g(a(\tau))|^2 \geqq |\operatorname{grad} W_g(b)|^2 - L\eta(\Delta t),$$

where L is the Lipschitz constant for the function $|\operatorname{grad} W_g(\cdot)|^2$ defined on $B_{2\eta(\Delta t)}(b)$.

We choose $\Delta t > 0$ so that

$$2L\eta(\Delta t) = \frac{1}{2} \min\{\sigma_0^2, \delta^{2\gamma}\}.$$

Then we conclude that the left-hand side of (5.15) is bounded below by $\Delta t \cdot \frac{1}{4} \min\{\sigma_0^2, \delta^{2\gamma}\} > 0$. This clearly contradicts (5.15).

We have therefore proved that there are two positive constants δ_0 and ε_0 so that either $\delta(\varepsilon) \geqq \delta_0$ or (i) is true whenever $0 < \varepsilon \leqq \varepsilon_0$. In particular, there is a critical point $a \in F$ of $W_g(\cdot)$ such that $|a - b| = \operatorname{dist}(b, F) \leqq C(\varepsilon, \delta(\varepsilon))$.

Next, for any critical point $\tilde{a}$ of $W_g(\cdot)$, we let $F(\tilde{a})$ be the connected component of F containing the point $\tilde{a}$. Since F is analytic, it is obvious that $W_g(\tilde{a}) = W_g(y)$

for all $y \in F(\tilde{a})$. Moreover, there is a small positive constant σ, depending on $W_g(\cdot)$, such that for any $\tilde{a}_1, \tilde{a}_2, \in F$, either $F(\tilde{a}_1) = F(\tilde{a}_2)$ or $\operatorname{dist}(F(\tilde{a}_1), F(\tilde{a}_2)) \geqq \sigma_1$ (see [8], sections 3.4.5–3.4.12).

We choose δ_0, ε_0 suitably small so that

$$\delta_0 \leqq \frac{1}{4\pi} \min\left\{\sigma_0^2, \left(\frac{\sigma_1}{3}\right)^{2\gamma}\right\}\Delta t \quad \text{and} \quad C(\varepsilon, \delta(\varepsilon)) \leqq \frac{\sigma_1}{3}$$

whenever $\varepsilon \in (0, \varepsilon_0)$ and $\delta_{(\varepsilon)} \in (0, \delta_0)$ where $\Delta t > 0$ is a number so that

$$2L^* \eta(\Delta t) = \frac{1}{2} \min\left\{\sigma_0^2, \left(\frac{\sigma_1}{3}\right)^{2\gamma}\right\},$$

$\eta : \mathbb{R}_+ \to \mathbb{R}$ is a strictly monotonic function such that $\eta(0^+) = 0$, and L^* is the Lipschitz norm of the function $|\operatorname{grad} W_g(\cdot)|^2$ on a $\frac{2}{3}\sigma_1$-neighborhood of $F(a)$.

For such choices of ε_0 and δ_0, we have both (i) and (ii) in Corollary 5.5. Indeed, the point b has to be in a $\frac{\sigma_1}{3}$-neighborhood of $F(a)$ because the number $C(\varepsilon, \delta(\varepsilon)) \leqq \frac{\delta_1}{3}$. Let $\bar{a}$ be as defined in Corollary 5.5. Then $\bar{a} \in F$ by [2], chapter 7. To show $\bar{a} \in F(a)$, it suffices to show $\bar{a}$ lies in a $\frac{2\sigma_1}{3}$-neighborhood of $F(a)$.

The latter fact follows from (5.4) and from the fact that the $A^*(\tau)$, $0 \leqq \tau < \infty$, are contained in a $\frac{2\delta_1}{3}$-neighborhood of $F(a)$. Otherwise, we would again find a sequence of $\varepsilon_n \downarrow 0$ and a sequence of initial data $u_0^{\varepsilon_n}$ (satisfying Assumptions 3.1 and 3.2 with $b_n = (b_1^n, \dots, b_d^n)$ in a $\frac{\delta_1}{3}$-neighborhood of $F(a)$) such that there would be a sequence of time $\{\tau_n\}$ for which the corresponding

$$u_{\varepsilon_n}(\cdot, \tau_n) \to \prod_{j=1}^{d} \frac{x - \bar{b}_j}{|x - \bar{b}_j|} e^{ih}$$

with $\bar{a} = (\bar{b}_1, \dots, \bar{b}_d)$, $\operatorname{dist}(\bar{b}, F(a)) = \frac{\sigma_1}{2}$. Then, as a result of the above arguments, we would conclude that

$$\begin{aligned}
&\varliminf_{n \to \infty} E_{\varepsilon_n}(u_{\varepsilon_n}(\cdot, \tau_n) - E_{\varepsilon_n}\left(u_{\varepsilon_n}(\cdot, \tau_n + \Delta t)\right) \\
&\geqq \frac{1}{\pi} \int_{\tau_n}^{\tau_n + \Delta t} \inf_{a(\tau) \in A^*(\tau)} |\operatorname{grad} W_g(a(\tau))|^2 \, d\tau \\
&> \frac{1}{4\pi} \min\left\{\sigma_0^2, \left(\frac{\sigma_1}{3}\right)^{2\gamma}\right\}\Delta t \geqq \delta_0 .
\end{aligned}$$

This result obviously contradicts our choices of $\delta_0, \Delta t$, and so on. We have thus completed the proof of Corollary 5.5.

6. Final Remarks

In conclusion, we would like to point out that the results in this paper can be generalized to the gradient flow of the energy functionals of the form

$$\int_\Omega \left[\frac{1}{2} |\nabla u|^2 + \frac{W(x)}{4\varepsilon^2} (|u|^2 - 1)^2\right] dx \equiv E_w^\varepsilon(u) \tag{6.1}$$

where $W(x)$ is a smooth, strictly positive function defined on Ω.

For minimizers of (6.1), the generalizations of [2] to this case has been carried out in recent preprints [13] and [16]. Both of these two works seem to prove only the following conclusions. There are a sequence $\varepsilon_n \downarrow 0$ and a sequence of minimizers of (6.1) $\{u_{\varepsilon_n}\}$, with $u_{\varepsilon_n} = g$ on $\partial\Omega$, such that $\{u_{\varepsilon_n}\}$ converges to

$$u_*(x) = \prod_{j=1}^{d} \frac{x - a_j}{|x - a_j|} e^{ih_a(x)}, \quad u_*|_{\partial\Omega} = g, \tag{6.2}$$

in the space $L^2(\Omega) \cap H'_{\text{loc}}(\overline{\Omega}\backslash\{a_1, \dots, a_d\})$ where $d = \deg(g, \partial\Omega), a_1, \dots, a_d$ are d distinct points so that $a = (a_1, \dots, a_d)$ minimizes certain renormalized energy $W(g, W, b), b \in \Omega^d$ (see [16]]). Here $\Delta h_a = 0$ in Ω.

Here we would like to add a few further remarks pertaining to the works of [13] and [16].

Remark 6.1. If u_ε is a minimizer of (6.1) with $u_\varepsilon = g$ on $\partial\Omega$, then $u_\varepsilon \in S_g(\lambda, K)$ for some positive constants λ, K depending only on g, Ω and the function $W(x)$. Indeed, if we use $\tilde{u}_\varepsilon$, which is a minimizer of (6.1) with $W(x) \equiv 1$, as a comparison function, then

$$E_W^\varepsilon(u_\varepsilon) \leqq E_W^\varepsilon(\tilde{u}_\varepsilon) \leqq \pi d \log \frac{1}{\varepsilon} + K. \tag{6.3}$$

Also, it is obvious that $|\nabla u_\varepsilon(x)| \leqq C/\varepsilon$ for all $x \in \Omega$ and for a constant $C = C(g, \Omega, W)$.

Since $u_\varepsilon \in S_g(\lambda, K)$, we may apply Theorem 2.4 and Lemma 4.1 to u_ε. It is clear that $N_\varepsilon = d$ in the statement of Theorem 2.4 for u_ε. It is also clear from Lemma 4.1 that, for any sequence $\varepsilon_n \downarrow 0$, there is a subsequence of $\{u_{\varepsilon_n}\}$ that converges to a map of the form u_* in (6.2). The fact that point $a = (a_1, \dots, a_d)$ is a global minimum point of $W(g, w, b)$ follows from a simple argument as in [16]].

Remark 6.2. When Ω is a smooth domain, we have

$$\int_\Omega \left(|u_\varepsilon|^2 - 1\right)^2 dx \leqq C(g, \Omega, W)\varepsilon^2 \tag{6.4}$$

as a minimizer u_ε of (6.1) with $u_\varepsilon = g$ on $\partial\Omega$.

To show (6.4), we let $\overline{u}_\varepsilon$ be a minimizer of

$$\int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{M}{4\varepsilon^2}\left(|u|^2 - 1\right)^2\right] dx, \ u|_{\partial\Omega} = g,$$

and let $\underline{u}_\varepsilon$ be a minimizer of

$$\int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{m}{4\varepsilon^2}\left(|u|^2 - 1\right)^2\right] dx, \ u|_{\partial\Omega} = g,$$

where $m = \frac{1}{2}\min_\Omega W(x), M = 2\max_\Omega W(x)$. Then it is clear that

$$\begin{aligned}
&\int_\Omega \left[\frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{W(x)}{4\varepsilon^2}\left(|u_\varepsilon|^2 - 1\right)^2\right] dx \quad (u_\varepsilon \text{ is a minimizer}) \\
&\leqq \int_\Omega \left[\frac{1}{2}|\nabla \bar{u}_\varepsilon|^2 + \frac{W(x)}{4\varepsilon^2}\left(|\bar{u}_\varepsilon|^2 - 1\right)^2\right] dx \\
&\leqq \int_\Omega \left[\frac{1}{2}|\nabla \bar{u}_\varepsilon|^2 + \frac{M}{4\varepsilon^2}\left(|\bar{u}_\varepsilon|^2 - 1\right)^2\right] dx \\
&\leqq \pi d \log\frac{1}{\varepsilon} + C(M, g, \Omega),
\end{aligned} \tag{6.5}$$

and also that

$$\begin{aligned}
\pi d \log\frac{1}{\varepsilon} - C(m, g, \Omega) &\leqq \int_\Omega \left[\frac{1}{2}|\nabla \underline{u}_\varepsilon|^2 + \frac{m}{4\varepsilon^2}\left(|\underline{u}_\varepsilon|^2 - 1\right)\right] dx \\
&\leqq \int_\Omega \left[\frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{m}{4\varepsilon^2}\left(|u_\varepsilon|^2 - 1\right)\right] dx \\
&\leqq \int_\Omega \left[\frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{W(x)}{4\varepsilon^2}\left(|u_\varepsilon|^2 - 1\right)\right] dx.
\end{aligned} \tag{6.6}$$

Combining (6.5) and (6.6) yields

$$\int_\Omega \frac{W(x) - m}{4\varepsilon^2}\left(|u_\varepsilon|^2 - 1\right)^2 dx \leqq C(m, g, \Omega) + C(M, g, \Omega). \tag{6.7}$$

Finally, (6.7) implies that (6.4) is true with $C(g, \Omega, w) \leqq [C(m, g, \Omega) + C(M, g, \Omega)] \cdot \frac{4}{m}$.

We also note that if Ω is simply connected, one may use Riemann mapping to transform Ω into a ball B. Of course, the function $W(x)$ may also change, but it remains strictly positive and bounded. In this case, we do not have to use the hard estimate (cf. [2])

$$\pi d \log\frac{1}{\varepsilon} - C(m, g, \Omega) \leqq \int_\Omega \left[\frac{1}{2}|\nabla \underline{u}_\varepsilon|^2 + \frac{m}{4\varepsilon^2}\left(|\underline{u}_\varepsilon|^2 - 1\right)^2\right] dx$$

in (6.6). Instead, we have

$$\begin{aligned}
&\int_\Omega \left[\frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{W(x)}{4\varepsilon^2}\left(|u_\varepsilon|^2 - 1\right)^2\right] \\
&\qquad \leqq \int_\Omega \left[\frac{1}{2}\left|\nabla \underline{u}_\varepsilon\right|^2 + \frac{W(x)}{4\varepsilon^2}\left(|\underline{u}_\varepsilon|^2 - 1\right)^2\right] dx.
\end{aligned} \tag{6.8}$$

Now, combining (6.8) and half of (6.6), we obtain that

$$\frac{1}{4\varepsilon^2}\int_\Omega (W(x) - m)\left(|u_\varepsilon|^2 - 1\right)^2 dx \leqq \frac{1}{4\varepsilon^2}\int_\Omega (W(x) - m)\left(|\underline{u}_\varepsilon| - 1\right)^2 dx.$$

Since we may assume here that Ω is a ball, and since $\int_\Omega \frac{m}{\varepsilon^2}(|\underline{u}_\varepsilon|^2-1)^2dx \leqq C(g,\Omega)$ is valid from the simple Pokhozhaev identity (see [2], III.3]), we see that (6.4) is true.

Remark 6.3. In this paper we do not discuss the cases that $u_\varepsilon^0(x)$ may have high-degree vortices or that the assumption $E_\varepsilon(u_\varepsilon^0) \leqq \pi d \log \frac{1}{\varepsilon} + K$ may not be valid. It will be very interesting to see whether a vortex of degree $d > 1$ will immediately split into d vortices of degree 1 in the time evolution (1.1)–(1.3).

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