Research Statement

Philip P. Mummert

Background and motivating questions

My primary research is in the area of discrete dynamical systems (iteration), specifically in the context of several complex variables. Consequently I have concomitant interests in general dynamical systems, one-variable complex dynamics, complex analysis, analysis of several complex variables, and computer programming. Dynamical systems play an increasingly important role in our mathematical understanding of the world, with applications to planetary motion, mechanics, finance, biology, and the long-term behavior of any system that evolves over time.

In particular, complex Hénon maps are fundamental to understanding higher-dimensional dynamics as a simple family of invertible functions that exhibit a wide range of dynamical phenomena. Primary questions concern the transition from order to chaos, structural stability, and developing and understanding relevant computer algorithms. Hénon maps have been explored by mathematicians and physicists alike, often in a real-valued setting. For a polynomial $p : \mathbb{C} \to \mathbb{C}$ and a parameter $b \in \mathbb{C} \setminus \{0\}$, consider the complex Hénon map $H : \mathbb{C}^2 \to \mathbb{C}^2$ given by:

$$H \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} w \\ p(w) - bz \end{pmatrix}.$$

Our interest is to understand the dynamics of this diffeomorphism under iteration, which generates the following very broad inquiries: What are simple models and classifications for the dynamics of H? How do the dynamics and the Julia set vary with p and b? What computer models and algorithms can calculate the relevant data?

It is known [HO] that for hyperbolic p with connected Julia set in $\mathbb{C}$, the dynamics of H and p are essentially the same if $|b| < \epsilon$. *(Question A) How big is ϵ? Can the perturbation from the one-dimensional map to the two-dimensional map be followed in a calculated, constructive way?* It is also known that H is simply the full two-shift, a "horseshoe map," for quadratic $p(w) = w^2 + c$, where $|c|$ is large. *(Question B) Can we be more precise about the region of parameter space for which H is a horseshoe? What two-shift automorphisms are induced by mondromy in the horseshoe parameter locus?* Looking at the dynamics outside the Julia set, when H is hyperbolic and unstably connected, the Bedford-Smillie [BS] conjugating map to the exterior solenoid is analogous to the bijective Böttcher map for one variable polynomials with connected Julia set. *(Question C) Is this solenoid map injective? How does it vary with p and b?*

Summary of primary research

In [M1] it is shown that the McMullen-Sullivan holomorphic motion [MS] for topologically conjugate, complex polynomials with connected Julia set in $\mathbb{C}$ follows level sets of the Böttcher coordinate. Analogously, the Buzzard-Verma holomorphic motion [BV] for hyperbolic, unstably connected, polynomial diffeomorphisms of $\mathbb{C}^2$ follows level sets of the

Bedford-Smillie solenoid map. It follows that this solenoid map is injective for those Hénon maps that are perturbations of (one-dimensional) hyperbolic maps with connected Julia set, which is a partial answer to *Question C*.

In [M2] a correcting operator for pseudo-orbits of the quadratic complex Hénon map is presented in response to *Question A*. In some settings the iterates of the operator converge to a holomorphic motion that respects the dynamics. This computational procedure is applied in the context of horseshoes and solenoids, and lends itself nicely to a computer algorithm for finding the Julia set of H: begin with a horseshoe/solenoid model and implicitly follow it through parameter space, by successively finding fixed points of a shadowing operator on orbit sequences. This procedure also gives a combinatorial description of the desired Julia set. Note that this method breaks down (presumably) at the boundary of the horseshoe/solenoid region, which, if true, might give an algorithmic means of finding the boundary. This shadowing may also lend deeper insight into a physicist's approach to finding periodic points of Hénon maps [BW].

More work remains in forming simple dynamical models and describing how they vary with parameters. The implicit shadowing of pseudo-orbits needs to be investigated further in this regard, especially with regard to symbolic dynamics inherited from the horseshoe or the solenoid. Also, how do the holomorphic motions described above extend to larger domains?

Recently I have also been investigating another algorithm based on a topological lifting that results from extending pseudo-orbits. That is to say, just as a one-variable Julia set can be calculated by successively lifting preimages of a large circle, in two variables one begins with a large torus and lifts paths to create longer orbit sequences which converge to the Julia set for the Hénon map. The nature of the lifting equips the Julia set with a topological model derived from the solenoid. This algorithm has the potential to combinatorially describe when the Julia set is connected and the manner in which it disconnects as parameters vary. It seems to be effective in programming practice, but more importantly, I believe the ideas behind the algorithm itself have theoretical implications for attempting to classify parameter space. In fact the implicit topological model is closely connected to iterated monodromy groups as well as solenoidal external rays. My research will attempt to clarify these connections.

Here is another project for the future in response to *Question B*. By the two-shift, we mean the dynamical system on the space of binary sequences given by shifting (e.g. $11010011\ldots \mapsto 1010011\ldots$). The Julia set for $z^2 + c$ is conjugate to the two-shift for large c values. Hence there is a bijection from points in the Julia set to binary sequences. Following the Julia set as we vary the parameter c about a large circle, Re^{it}, which encompasses the Mandelbrot Set, upon reaching $t = 2\pi$ we see that each point of the original Julia set has landed at another point, inducing an automorphism of the two-shift. In fact, it is the automorphism where we switch all the 0's and 1's. Topologically, we can deduce that since the automorphism is non-trivial, we must have encircled a region of parameter space for which not all Julia sets are conjugate to the two-shift. Turning our attention to two complex variables, the quadratic complex Hénon map is given by $H(z, w) = (w, w^2 + c - bz)$. For small $|b|$ values, this map is merely a two-dimensional perturbation of the one-dimensional quadratic map. For non-zero b values the map is invertible and for large $|c|$ values the Julia set of H is conjugate to the *full* two-shift, where now we consider bi-infinite binary sequences (e.g. $\ldots 101.101\ldots \mapsto$

...1011.01...). The nature of the topology of the "Mandelbrot set" (parameter space) for the Hénon map is unclear. Following the Julia set as we trace a loop of parameter values induces an element of the mysterious group of automorphisms of the full two-shift. Possible goals for this project: (1) Develop algorithms that will follow specific points (i.e. periodic points) in the Julia set as we change the parameters. (Can we prove it works?) (2) Find loops in parameter space that induce non-trivial automorphisms of the full two-shift. (3) Attempt to catalog the possibilities, i.e. what automorphisms are possible?

Undergraduate research projects

While a graduate student I assisted with the 2006 Purdue Research Experience for Undergraduates (REU) in the numerical computation of zeroes of holomorphic functions. As a professor at Taylor University, I received an internal grant in 2008, 2009, 2010, and 2012 from its Center for Research and Innovation to mentor undergraduates in summer mathematical research. In academic year 2014-5 I received a CURM grant (from BYU/NSF) for a substantial undergraduate research project at Butler University related to symbolic dynamics. One of my previous projects investigated the long-term behavior for a particular type of finite one-dimensional cellular automata. Our work tied in nicely with a previous faculty-undergraduate collaboration at another institution [SMCB]. Another project considered a discretized version of complex analysis on an integer lattice of the plane [SM]. Here, the basic notion is to replace the Cauchy-Riemann (differential) equations, with analogous discrete (difference) equations. Many of the results of classical, (continuous) complex analysis have recognizable analogues in a discrete setting. There are multiple approaches to addressing this question, and similar questions have been explored before under the monikers of "preholomorphic functions" and "monodiffric functions of the first kind." The theory of finite (or discrete) calculus, i.e. finite differences, has been well established and includes a unified theory of time scales that encompasses both continuous and discrete calculus (for real variables). The situation seems murkier in the complex plane and in my opinion, the literature in this area seems to be unorganized. I would like to help undergraduates investigate these concepts further in the future.

Turning to a dynamical system on integer partitions introduced in [AD], "Austrian Solitaire" begins with a deck of cards of size n and lays out piles of size no greater than L including one special pile called the bank. To play a round, remove one card from each ordinary pile, place them in the bank, and form as many new piles of size L from the bank as possible. Although I've described it here using cards, it also has a natural economics interpretation. It was conjectured that the original configuration of piles is irrelevant to the long-term proceedings; in other words, the resulting cycle is completely determined by n and L regardless of the initial state of the cards. I recently proved this conjecture, and can characterize the resulting cycles with a modified Farey sequence. Some of the related implications for accounting, dinner table problems, certain physics questions, and music theory would be fun to investigate with undergraduates in the future.

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