Secondary stability and periodicity for unordered configuration spaces

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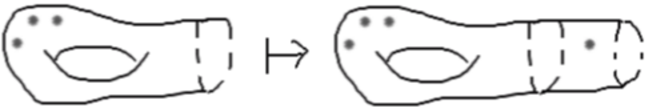
Outline

- Homological stability
- Secondary stability
- Periodic stability

Homological Stability for Unordered Configuration Spaces

Let M be a connected open manifold (=not compact).
$\mathrm{Conf}_k(M) := \{\{m_1, \ldots, m_k\} \subset M : m_i \neq m_j \text{ for i } \neq \text{j}\}$.
Since M is open, there is a map (called the **stabilization map**)

$$\sigma : \mathrm{Conf}_k(M) \to \mathrm{Conf}_{k+1}(M).$$



Theorem (McDuff)

Let M be a connected open manifold. The stabilization map $\sigma : Conf_k(M) \to Conf_{k+1}(M)$ induces an isomorphism in integral homology

$$\sigma_* : H_i(Conf_k(M)) \to H_i(Conf_{k+1}(M))$$

for $k \gg i$.

Secondary Stability

- First instance of secondary stability due to work of Galatius-Kupers-Randal–Williams on mapping class groups.
- Homological degree is not fixed.
- Measures unstable homology groups $H_i(\mathrm{Conf}_k(M))$ by checking if $H_*(\mathrm{Conf}_{k+1}(M), \mathrm{Conf}_k(M))$ stabililze in some way.

Theorem (H.)

Let M be a connected open surface. There is a map $s_2 : H_i(Conf_{k+1}(M), Conf_k(M); \mathbb{F}_p) \to H_{i+2p-2}(Conf_{k+1+2p}(M), Conf_{k+2p}(M); \mathbb{F}_p)$ which is an isomorphism for $k > \frac{p^2}{p^2-1} i + \frac{2p^2-2p-2}{p^2-1}$.

Periodic Stability

Question: Is there homological stability when a manifold M is closed (=compact and without boundary)?

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Two issues:

1) If M is closed, there does not exist an obvious stabilitization map

$$\sigma : \mathrm{Conf}_k(M) \to \mathrm{Conf}_{k+1}(M).$$

2) $H_1(\mathrm{Conf}_k(S^2); \mathbb{Z}) \cong \frac{\mathbb{Z}}{2k-2}$ for $k \geq 2$ (due to Fadell-Van Buskirk).

Theorem (Cantero-Palmer, Nagpal, Kupers-Miller)

Let M be a connected closed manifold. Then $H_i(Conf_k(M); \mathbb{F}_p) \cong H_i(Conf_{k+p}(M); \mathbb{F}_p)$ for $k \gg i$.

Periodic Secondary Stability

By constructing a stabilization map on the chain level $C_*(\mathrm{Conf}(M); \mathbb{F}_p)$ (instead of on the space level) for a closed manifold M, we obtain periodic secondary stability for a closed surface.

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Theorem (H.)

Let M be a connected ~~open~~ closed surface. There is a map $s_2 : H_i(Conf_{\cancel{k+1}\ k+p}(M), Conf_k(M); \mathbb{F}_p) \to H_{i+2p-2}(Conf_{\cancel{k+1}\ k+p+2p}(M), Conf_{k+2p}(M); \mathbb{F}_p)$ which is an isomorphism for $k > \frac{p^2}{p^2-1} i + \frac{2p^2-2p-2}{p^2-1}$.

Proof Sketch for Periodic Secondary Stability

Let M be a closed surface and $D \subset M$ an open disk in M.

- $\mathrm{Conf}(M) \simeq |B_\bullet(\mathrm{Conf}(M \setminus \bar{D}), \mathrm{Conf}(S^1 \times [0,\infty)), \mathrm{Conf}(D))|$.
- Should think of $\mathrm{Conf}(S^1 \times [0,\infty))$ as a monoid and $\mathrm{Conf}(D)$ as a module over $\mathrm{Conf}(S^1 \times [0,\infty))$.
- On the RHS, it is enough to construct a stabilization map for $C_*(\mathrm{Conf}(D); \mathbb{F}_p)$ that preserves the $\mathrm{Conf}(S^1 \times [0,\infty))$-module structure of $\mathrm{Conf}(D)$.