Lesson 10

Exact Equations (2.6)

Consider the differential equation
$$(2xy^2+2y)+(2x^2y+2x)y'=0$$

Is this equation linear? No! (integrating factors doesn't work)
Is it separable? No!

But we can notice something very interesting
Consider the function $\psi(x,y)=x^2y^2+2xy$.

$\frac{\partial\psi}{\partial x}=\psi_x=2xy^2+2y$ and $\frac{\partial\psi}{\partial y}=\psi_y=2x^2y+2x$

Knowing this, the diff eq above is of the form
$$\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\cdot\frac{dy}{dx}=0$$

Since $\psi(x,y)$ is a function of x and y, the (multivariate) chain rule tells us that
$$\frac{d\psi}{dx}=\frac{\partial\psi}{\partial x}\cdot\underbrace{\frac{dx}{dx}}_{=1}+\frac{\partial\psi}{\partial y}\cdot\frac{dy}{dx}$$

So really, our differential equation is of the form
$$\frac{d\psi}{dx}=0$$

So we get $\psi(x,y)=C$

Thus, $\boxed{x^2y^2+2xy=C}$ implicitly is a solution!

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A diff eq of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is called exact if there exists a function $\psi(x,y)$ with $\psi_x(x,y) = M(x,y)$ and $\psi_y(x,y) = N(x,y)$.

The solutions of an exact equation can be given implicitly by $\psi(x,y) = c$ (c an arbitrary constant)

Theorem 2.6.1 If M, N, M_y, N_x are continuous in the rectangular region $R: \alpha < x < \beta, \gamma < y < \delta$, then

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact on R (i.e., there exists a function $\psi(x,y)$ with $\psi_x = M$ and $\psi_y = N$) if and only if $M_y = N_x$.

Ex 1. Determine whether the following differential equations are exact:

(a) $\underbrace{(e^x \sin y + 2xy)}_{M(x,y)} + \underbrace{(e^x \cos y + x^2)}_{N(x,y)}\frac{dy}{dx} = 0$

$M_y = e^x \cos y + 2x$ $\qquad$ $N_x = e^x \cos y + 2x$

$$M_y = N_x$$

so by Thm 2.6.1, the equation is exact.

(b) $(3x^2+y) - (2y+x)\,y' = 0$

Careful here! $M(x,y) = 3x^2+y$, $N(x,y) = -2y-x$

$M_y = 1$, $N_x = -1$

Since $M_y \neq N_x$, the equation is not exact

(c) $y' = \dfrac{6y+2x}{3y^2-6x}$

$(3y^2-6x)\,y' = 6y+2x$

$\underbrace{(-6y-2x)}_{M} + \underbrace{(3y^2-6x)}_{N}\,y' = 0$

$M_y = -6$, $N_x = -6$

$M_y = N_x$, so the equation is exact.

So, if we have an exact equation, how can we figure out what $\Psi(x,y)$ is?

Well, we know $\Psi_x(x,y) = M(x,y)$ and $\Psi_y(x,y) = N(x,y)$.

If we integrate $M(x,y)$ with respect to x, we should get $\Psi(x,y)$ (almost... up to a function of y). We can then differentiate our result with respect to y and that should equal $N(x,y)$. By using another integration, we can finally find $\Psi(x,y)$ exactly.

Ex 2. Solve $(e^x \sin y + 2xy) + (e^x \cos y + x^2) y' = 0$.

In example 1, we checked that this is exact.
Thus, there exists a function $\Psi(x,y)$ such that
$\Psi_x(x,y) = M(x,y) = e^x \sin y + 2xy$ and
$\Psi_y(x,y) = N(x,y) = e^x \cos y + x^2$

$$\Psi(x,y) = \int \Psi_x(x,y)\,dx = \int M(x,y)\,dx = \int (e^x \sin y + 2xy)\,dx$$

$$= e^x \sin y + x^2 y + \underbrace{h(y)}_{\text{some function of } y}$$

Since $\Psi(x,y) = e^x \sin y + x^2 y + h(y)$,
$\Psi_y(x,y) = e^x \cos y + x^2 + h'(y)$

But from above, $\Psi_y(x,y) = N(x,y) = e^x \cos y + x^2$
so $e^x \cos y + x^2 + h'(y) = e^x \cos y + x^2$
$\Rightarrow \quad h'(y) = 0$
$h(y) = \int h'(y)\,dy = \int 0\,dy = 0 + C_1$ ← (This constant doesn't matter!)

Thus, $\Psi(x,y) = e^x \sin y + x^2 y + h(y) = e^x \sin y + x^2 y + C_1$

Hence, $e^x \sin y + x^2 y + C_1 = C_2$
so $e^x \sin y + x^2 y = C_2 - C_1$ ← just a constant

so $\boxed{e^x \sin y + x^2 y = C}$

is a solution.

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Ex 3. Solve $y' = \dfrac{6y+2x}{3y^2-6x}$

In example 1, we saw we could rewrite this as $\underbrace{(-6y-2x)}_{M} + \underbrace{(3y^2-6x)}_{N} y' = 0$

and it is exact.

Thus, there exists a function $\psi(x,y)$ such that $\psi_x = -6y-2x$ and $\psi_y = 3y^2-6x$

$\psi(x,y) = \int(-6y-2x)\,dx = -6yx - x^2 + h(y)$

so

$\psi_y = -6x + h'(y)$

But by above $\psi_y = 3y^2 - 6x$

so $3y^2 - 6x = -6x + h'(y)$.

Thus, $h'(y) = 3y^2$

$h(y) = \int 3y^2\,dy = y^3$ (constant doesn't matter)

$\psi(x,y) = -6yx - x^2 + h(y) = -6yx - x^2 + y^3$

Hence $\boxed{-6yx - x^2 + y^3 = C}$ is a solution

Ex 4. Solve the IVP and determine where the solution is valid.

$$\underbrace{(2x-y)}_{M} + \underbrace{(2y-x)}_{N}y' = 0, \quad y(1)=3$$

exact? $M_y = -1$, $N_x = -1$, so yes!

$\psi_x = 2x - y$, $\psi_y = 2y - x$

$$\psi(x,y) = \int (2x-y)\,dx = x^2 - yx + h(y)$$

so $\psi_y = -x + h'(y)$

$$2y - x = -x + h'(y)$$

$$2y = h'(y)$$

$$h(y) = \int 2y\,dy = y^2$$

$$\psi(x,y) = x^2 - yx + y^2$$

$$x^2 - yx + y^2 = C$$

Since $y(1) = 3 \ldots \quad 1 - 3 + 3^2 = C = 7$

$$y^2 - xy + x^2 - 7 = 0$$

This is a quadratic in y.
Quadratic formula gives

$$y = \frac{-(-x) \pm \sqrt{(-x)^2 - 4(1)(x^2-7)}}{2(1)}$$

$$y = \frac{x \pm \sqrt{x^2 - 4x^2 + 28}}{2}$$

$$y = \frac{x \pm \sqrt{28 - 3x^2}}{2}$$

In order to satisfy the initial condition,

$$3 = \frac{1 \pm \sqrt{28-3}}{2} = \frac{1 \pm 5}{2} \quad \text{requires } +$$

so $\boxed{y = \frac{1}{2}\left(x + \sqrt{28-3x^2}\right)}$

This is valid when $28 - 3x^2 \geq 0$

$$28 \geq 3x^2$$

$$\frac{28}{3} \geq x^2$$

$$\sqrt{28/3} \geq |x|$$

However, if $x = \pm\sqrt{\frac{28}{3}}$

$$y = \pm\frac{1}{2}\sqrt{\frac{28}{3}}$$

Looking back at the original diff eq...

$$\left(2\left(\pm\sqrt{\tfrac{28}{3}}\right) \mp \tfrac{1}{2}\sqrt{\tfrac{28}{3}}\right) + \underbrace{\left(2\left(\pm\tfrac{1}{2}\sqrt{\tfrac{28}{3}}\right) \mp \sqrt{\tfrac{28}{3}}\right)}_{=0} y' = 0$$

so the diff eq is not satisfied.

Thus, the solution is valid

when $\boxed{|x| < \sqrt{\frac{28}{3}}}$

or equivalently, $\boxed{-\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}}}$