Eddie Price - MA 266, Lesson 12 (SP 19)

Euler's Method (Part 2)

While Euler's Method is a very intuitive method, it has some major downsides. In order to be remotely accurate, you may need to have very small values of h; however, the smaller the value of h we use, the more computationally expensive Euler's Method is. This is a downside.

On your homework for this lesson, you will have some problems that explore the downsides to Euler's Method.

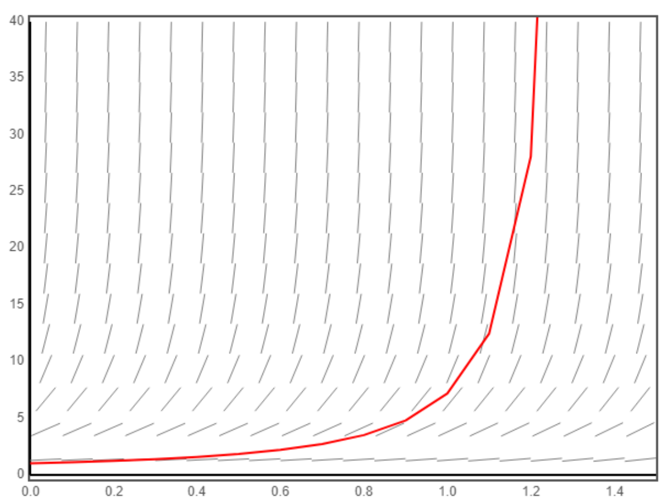
Example 1. Consider the IVP $y' = t^2 + y^2$, $y(0) = 1$. Use Euler's method with $h = 0.1, 0.05, 0.025$, and 0.01 to explore the solution of this problem for $0 \leq t \leq 1$. What is your best estimate of the value of the solution at $t = 0.8$? At $t = 1$? Are your results consistent with the direction field?

Using eul.m or an online Euler's method calculator, one gets the following information:

h	t = 0.8	t = 1
0.1	3.507832	7.189548
0.05	4.201315	12.320930
0.025	4.800433	23.926010
0.01	5.342756	90.755506

Based on the above data, one might guess that $y(0.8) \approx 6$. However, one could be very confused about $y(1)$, though one may suspect, based on the above data, that y has a vertical tangent at $t = 1$, since the outputs are growing at an increasing rate for smaller values of h.

Looking at the direction field, we see:



(from https://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html)
Note: the red curve is a numerical approximation, not the actual solution!

From the direction field, we can see that our estimate of $y(0.8)$ is pretty good. (If you use WolframAlpha to get $y(0.8)$, it gives ≈ 5.85). From the direction field, it seems plausible that the solution does have a vertical asymptote at $t = 1$.

The take away point here is that if something goes wrong with the domain of a function, Euler's method is not a terribly reliable tool. So you have to be wary. If you were to just assume you would get a really accurate result using $h = 0.01$, Euler's method would approximate $y(1)$ as about 90, and you may assume the value there is 90, even though the solution has an asymptote there. So we must be wary in using Euler's method when it comes to domain restrictions.

Example 2. If $y' = 7y - e^{-t}$, $y(0) = 1$, compute $y(1)$ explicitly. Then use Euler's method with step sizes $h = 1/n$ to find the smallest integer n which will give a value that approximates the explicit value of $y(1)$ correct to within 0.5.

Using the method of integrating factors, one obtains $y(t) = \frac{7}{8}e^{7t} + \frac{1}{8}e^{-t}$. Then $y(1) = \frac{7}{8}e^{7} + \frac{1}{8}e^{-1} \approx 959.599$. Hence, to be within 0.05, the approximation of $y(1)$ should be in the interval $[959.599 - 0.5, 959.599 + 0.5] = [959.099, 960.099]$.

You might try things like $n = 10$ at first and notice you are way off. Quickly, you try $n = 100$, then $n = 500$, then $n = 1000$, then $n = 5000$, then $n = 10,000$, then $n = 50,000$. Finally, we are in the proper range. With a bit of patience, one can determine that the smallest positive integer n which works is $n = 46,925$. Notice that $1/46925 \approx 0.000002$. As such, we need an incredibly small step size to be accurate here.

This being said, when using Euler's method, we have to be careful about step size when it comes to functions that grow quickly. You can't just assume that 0.01 will be good enough for your purposes.