Lesson 18

Variation of Parameters (3.6)

Variation of Parameters is another technique to solve nonhomogeneous second order linear equations. Unlike the method of Undetermined coefficients, no guessing is involved. We do have to integrate in this method, however.

Much like in Reduction of Order, we assume the solution to $y''+p(t)y'+q(t)y=g(t)$ is of the form $u_1(t)y_1(t)+u_2(t)y_2(t)$ where $\{y_1, y_2\}$ are a fundamental set of solutions to the homogeneous equation. The goal is now to find $u_1(t)$ and $u_2(t)$.

$$y(t) = u_1 y_1 + u_2 y_2$$
$$y'(t) = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$
$$= u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$$

Here, we will assume that the following equation is true

$$(*) \quad u_1' y_1 + u_2' y_2 = 0$$

This will make our calculations easier, since we will end up only having to solve for u_1' and u_2', instead of having terms involving the second derivatives.

Then

$$y' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$$
$$= u_1 y_1' + u_2 y_2'$$

Therefore, $y'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$

We plug this into $y'' + py' + qy = g$

$$\underline{u_1 y_1''} + u_1' y_1' + u_2 y_2'' + u_2' y_2' + p(\underline{u_1 y_1'} + u_2 y_2')$$
$$+ q(\underline{u_1 y_1} + u_2 y_2) = g$$

So... rearranging terms, we obtain...

$$u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2)$$
$$+ u_1' y_1' + u_2' y_2' = g$$

Since y_1 and y_2 are solutions to the homogeneous equation, $y_1'' + py_1' + qy_1 = 0$ and $y_2'' + py_2' + qy_2' = 0$, leaving us with

$$(**) \quad u_1' y_1' + u_2' y_2' = g$$

(*) and (**) form the system of equations

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 & (*) \\ u_1' y_1' + u_2' y_2' = g & (**) \end{cases}$$

From (*), we get $u_1' y_1 = -u_2' y_2$

So $u_1' = \dfrac{-u_2' y_2}{y_1}$

Plugging in to (**), we get $\left(-\dfrac{u_2' y_2}{y_1}\right) y_1' + u_2' y_2' = g$

$$u_2'\left(\frac{-y_2 y_1'}{y_1} + y_2'\right) = g$$

So $u_2'\left(\frac{-y_2 y_1' + y_1 y_2'}{y_1}\right) = g$

$y_1 y_2' - y_2 y_1' = W$, the Wronskian of y_1 and y_2.

so
$u_2'\left(\frac{W}{y_1}\right) = g$, meaning, $u_2' = \frac{y_1 g}{W}$.

Doing a similar process yields

$$u_1' = \frac{-y_2 g}{W}$$

We can find u_1 and u_2 by integrating.

<u>Method of Variation of Parameters</u>

Given an equation $y'' + p(t)y' + q(t)y = g(t)$

1. Make sure to write the equation in this form!
2. Find a fundamental set of solutions $\{y_1, y_2\}$ to the homogeneous equation.
3. Assume $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$.
4. Compute the Wronskian $W(t)$ of y_1 and y_2 (in that order).
5. $u_1(t) = -\int \frac{y_2(t)g(t)}{W(t)}\,dt + c_1$

   $u_2(t) = \int \frac{y_1(t)g(t)}{W(t)}\,dt + c_2$
6. Plug in $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ to obtain the general solution.

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Ex 1. Find the general solution
$$y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$$

1. In proper form! ✓ $\quad g(t) = t^{-2}e^{-2t}$
2. $r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r = -2$

   $y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

   $y_1(t) = e^{-2t}, \quad y_2(t) = te^{-2t}$
3. Assume $y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$.
4. $$W(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & -2te^{-2t} + e^{-2t} \end{vmatrix} = -2te^{-4t} + e^{-4t} + 2te^{-4t}$$

   so $\quad W(t) = e^{-4t}$
5. Then
$$u_1(t) = -\int \frac{y_2(t)\,g(t)}{W(t)}\,dt + c_1 = -\int \frac{te^{-2t} \cdot t^{-2}e^{-2t}}{e^{-4t}}\,dt + c_1$$
$$= -\int t^{-1}\,dt + c_1 = -\ln(t) + c_1$$
$$u_2(t) = \int \frac{y_1(t)\,g(t)}{W(t)}\,dt + c_2 = \int \frac{e^{-2t} \cdot t^{-2}e^{-2t}}{e^{-4t}}\,dt + c_2$$
$$= \int t^{-2}\,dt + c_2 = -t^{-1} + c_2$$
6. $$y(t) = (-\ln(t) + c_1)e^{-2t} + (-t^{-1} + c_2)te^{-2t}$$
$$= -e^{-2t}\ln(t) + c_1 e^{-2t} - e^{-2t} + c_2 te^{-2t}$$
$$= \underbrace{(c_1 - 1)}_{\text{just a constant}} e^{-2t} + c_2 te^{-2t} - e^{-2t}\ln(t)$$

so $$\boxed{y(t) = c_1 e^{-2t} + c_2 te^{-2t} - e^{-2t}\ln(t)}$$

Ex 2. Find a particular solution to
$t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \ t > 0$
where $y_1(t) = t$, $y_2(t) = te^t$ are solutions to the homogeneous equation.

We should check that y_1 and y_2 are a fundamental set of solutions.

$y_1' = 1, \ y_1'' = 0$
$t^2(0) - t(t+2)(1) + (t+2)t = 0$ ✓

$y_2' = te^t + e^t, \ y_2'' = te^t + 2e^t$
$t^2(te^t + 2e^t) - t(t+2)(te^t + e^t) + (t+2)(te^t) = 0$ ✓

$$W(t) = \begin{vmatrix} t & te^t \\ 1 & te^t + e^t \end{vmatrix} = t^2e^t + te^t - te^t = t^2e^t$$

1. $y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 2t, \ t > 0$
   $g(t) = 2t$
2. Already found $y_1(t) = t$, $y_2(t) = te^t$
3. Assume $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$
4. Already found $W(t) = t^2e^t$
5. $u_1(t) = -\int \frac{y_2(t)g(t)}{W(t)}dt + c_1 = -\int \frac{te^t \cdot 2t}{t^2e^t}dt + c_1$
   $= -\int 2dt + c_1 = -2t + c_1$
   $u_2(t) = \int \frac{y_1(t)g(t)}{W(t)}dt + c_2 = \int \frac{t \cdot 2t}{t^2e^t}dt + c_2$
   $= \int 2e^{-t}dt + c_2 = -2e^{-t} + c_2$
6. $y(t) = (-2t + c_1)t + (-2e^{-t} + c_2)te^t$
   $= -2t^2 + c_1t - 2t + c_2te^t$
   $= (c_1 - 2)t + c_2te^t - 2t^2$
   $= c_1t + c_2te^t - 2t^2$

So $\boxed{Y(t) = -2t^2}$ is a particular solution.