Lesson 21
Forced Vibrations

In this lesson, we assume there is an external force acting on the mass as well.

$$mg - \gamma u' - k(L+u) + \underbrace{g(t)}_{\text{external force}} = mu''$$

$$\boxed{mu'' + \gamma u' + ku = g(t)}$$

So now we get a nonhomogeneous diff eq. We refer to $g(t)$ as the forcing function. In this lesson, we deal only with periodic forcing functions; i.e., $g(t) = \cos(\omega t)$ or $\sin(\omega t)$.

Always be careful about units!

Ex 1. A mass of 2 kg stretches a spring 8 cm. The mass is acted on by an external force of $3\cos(2t)$ N and has a damping force which has magnitude 3N when the mass has a speed of 2 cm/s. If the mass is initially at rest in its equilibrium position. Set up an IVP.

$1\text{ N} = 1\ \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$. Notice Newtons implies we are working with kg, m, and s, so convert all units accordingly.

$m = 2\text{ kg}$, $L = 8\text{cm} = 0.08\text{m}$

Speed of $2\ \frac{\text{cm}}{\text{s}} = \frac{2\text{ cm}}{\text{s}} \cdot \frac{1\text{ m}}{100\text{ cm}} = 0.02\text{ m/s}$

$$mg - kL = 0$$

$$(2\text{ kg})(9.8\text{ m/s}^2) - k(0.08\text{m}) = 0 \Rightarrow k = 245\text{ kg/s}^2$$

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Damping force is $\gamma u'$, where u' is velocity.
If damping force is 3 N when velocity is 0.02 m/s,

$$\gamma(0.02) = 3 \Rightarrow \gamma = 150 \ \frac{N \cdot s}{m}$$

$$\boxed{\begin{array}{l} 2u'' + 150u' + 245u = 3\cos(2t),\ u(0)=0,\ u'(0)=0 \\ u \text{ is measured in meters, } t \text{ in seconds} \end{array}}$$

Damped, Forced Vibrations.

When the vibrations are damped, we have one of the following scenarios:

- $y_c(t) = c_1 e^{-rt} + c_2 t e^{-rt}$
- $y_c(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$
- $y_c(t) = c_1 e^{-\lambda t}\cos\mu t + c_2 e^{-\lambda t}\sin\mu t$

Using undetermined coefficients, our initial guess is $A\cos(\omega t) + B\sin(\omega t)$, and this cannot be in y_c.

Hence, solutions are of the form

$$y(t) = \underbrace{y_c(t)}_{\substack{\text{Transient} \\ \text{solution}}} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{\substack{\text{Steady} \\ \text{State solution}}}$$

"Transient" means only lasting for a short time and, in all cases, $\lim_{t\to\infty} y_c(t) = 0$.

So solutions are asymptotic to $A\cos(\omega t) + B\sin(\omega t)$
Eventually, the external force is all that matters!

(look at image on page 209 of textbook)

Undamped, Forced Vibrations

Here, $y_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$.

The particular solution then falls into one of two cases: (depending on the forcing frequency ω)

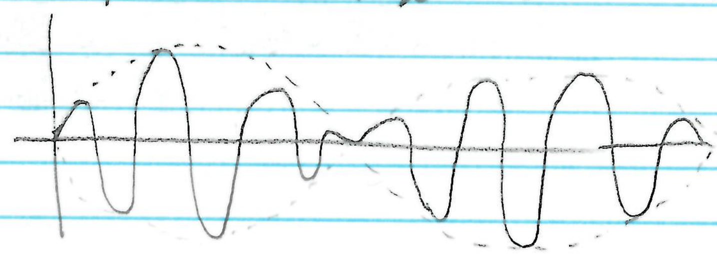
If $\omega \neq \omega_0$, $Y(t) = A\cos(\omega t) + B\sin(\omega t)$

If $\omega = \omega_0$, $Y(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$

$\underline{\omega \neq \omega_0}$: In this case, the solution is a sum of periodic motions with different amplitudes and/or frequencies.

This results in a graph that has almost erratic behavior, but the amplitudes grow and shrink in a periodic behavior.

(see picture on page 215 of textbook)



Such behavior is called a "beat" or "amplitude modulation"

"Beats" are used to tune musical instruments by ear – you want to eliminate beats so both (or all) instruments are playing at the same frequency.

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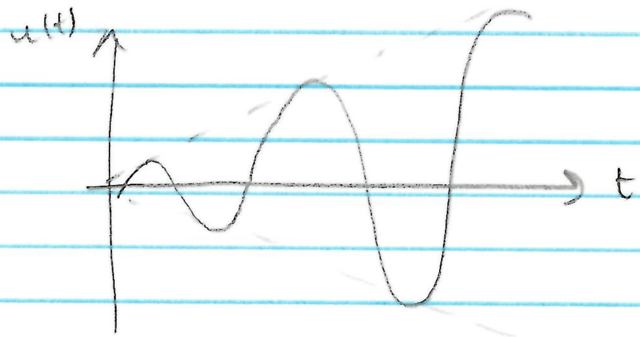
$\omega = \omega_0$: In this case, because the particular solution is $At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$, $y(t)$ has ubounded oscillations (see page 216 image)



In this situation ($\omega = \omega_0$), the system experiences resonance.
Resonance can be good (using seismographs to detect small earthquakes) or bad (see the collapse of the Tacoma Narrows Bridge).
Also, music can sound incredible when all musicians are performing in tune (resonance).

Ex 2. A mass weighing 3 lb stretches a spring 1 in. Assume there is no damping on the mass. An external force of $2\cos(\omega t)$ lb is applied. Determine the value of ω for which resonance occurs.

Unit of force here is lbs. So standard unit for distance is feet.

$mg = 3\text{ lbs}$, $g = 32\ ft/s^2$, so $m = \frac{3}{32}\ \frac{lb \cdot s^2}{ft} = \frac{3}{32}$ slugs

$L = 1\text{ in} = \frac{1}{12}\ ft.$

$$mg - kL = 0$$
$$3 \text{ lbs} - k\left(\tfrac{1}{12} \text{ ft}\right) = 0$$
$$k = 36 \text{ lb/ft}.$$

$$\frac{3}{32} u'' + 36u = 2\cos(\omega t)$$

Resonance occurs when $\omega = \omega_0$, the natural frequency, which is found in the complementary solution.

$$\frac{3}{32} r^2 + 36 = 0$$
$$r^2 = -384$$
$$r = \pm i\sqrt{384} = \pm i \cdot 8\sqrt{6}$$
$$u(t) = c_1 \cos(8\sqrt{6}\, t) + c_2 \sin(8\sqrt{6}\, t)$$
$$\omega_0 = 8\sqrt{6}$$

resonance occurs when external force has frequency $\boxed{\omega = 8\sqrt{6}}$