Lesson 24

The Laplace Transform (6.1)

Review of Improper Integrals

In this section, we deal with integrals of the form $\int_a^\infty f(t)\,dt$. Recall that these integrals are defined as

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty} \int_a^A f(t)\,dt$$

If the limit converges, the integral converges, and if the limit diverges, the integral diverges.

Ex 1. Find the values of s so that $\int_0^\infty e^{st}\,dt$ converges and find the value of the integral when it does.

$$\int_0^\infty e^{st}\,dt = \lim_{A\to\infty} \int_0^A e^{st}\,dt$$

$$= \lim_{A\to\infty} \left.\frac{e^{st}}{s}\right|_0^A$$

$$= \lim_{A\to\infty} \left[\frac{e^{sA}}{s} - \frac{1}{s}\right]$$

$$= \frac{\left(\lim_{A\to\infty} e^{sA}\right) - 1}{s}$$

This converges only when $\boxed{s<0}$ and converges to

$$\boxed{-\frac{1}{s}}$$

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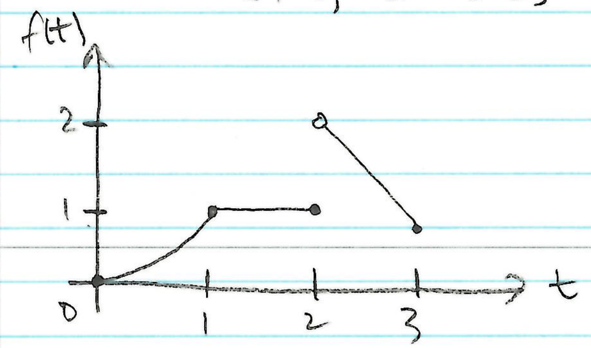
In order to take the integral of a function, we need the function to be mainly continuous. It doesn't have to be continuous everywhere, just almost everywhere. This leads to the notion of piecewise continuity:

Defn. A function $f(t)$ is piecewise continuous on an interval $\alpha < t < \beta$ (including possibly $\alpha = -\infty$ and/or $\beta = \infty$) if there exist finitely many points $t_1, \ldots, t_n$ in $\alpha < t < \beta$ such that $f(t)$ is continuous on $t_1 < t < t_2$, $t_2 < t < t_3$, $\ldots$, $t_{n-1} < t < t_n$ and that for all k, $\lim\limits_{t \to t_k^+} f(t)$ exists and $\lim\limits_{t \to t_{k+1}^-} f(t)$ exists.

In other words, $f(t)$ is piecewise continuous on $\alpha < t < \beta$ if $f(t)$ has at most finitely many discontinuities and each discontinuity is either a jump or removable discontinuity (no infinite discontinuities).

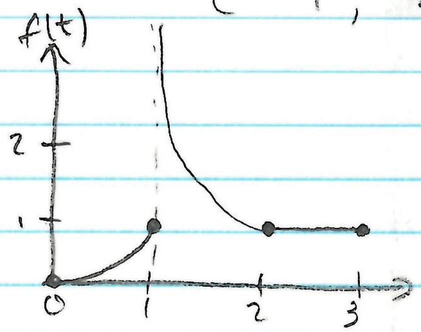
Ex 2. Are the following functions piecewise continuous?

(a) $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 4-t, & 2 < t < 3 \end{cases}$



yes, piecewise continuous on $0 < t < 3$.

(b) $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ (t-1)^{-1}, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$



not piecewise continuous since it has an infinite discontinuity

If a function $f(t)$ is piecewise continuous on $\alpha < t < \beta$ and its discontinuities occur at $t_1, \ldots, t_n$, then

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \cdots + \int_{t_n}^{\beta} f(t)\,dt$$

Definition. If $f(t)$ is piecewise continuous or continuous, the Laplace Transform of $f(t)$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

provided that the integral converges for some values of s.

The Laplace transform $F(s)$ is a function of s, but s is treated like a constant in the actual integral, since we are integrating with respect to t.

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Ex 3. Let $f(t) = e^{at}$ for a constant a.
Compute $\mathcal{L}\{f(t)\}$ and give the domain.

$$F(s) = \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \lim_{A\to\infty} \int_0^A e^{(-s+a)t}\, dt$$

$$= \lim_{A\to\infty} \left[\frac{e^{(-s+a)t}}{(-s+a)} \right]_{t=0}^{t=A}$$

$$= \lim_{A\to\infty} \left[\underbrace{\frac{e^{(-s+a)A}}{(-s+a)}}_{\substack{\text{by example 1, converges} \\ \text{if and only if } -s+a<0, \text{ i.e., } s>a}} - \frac{1}{(-s+a)} \right]$$

When $s>a$,

$$0 - \frac{1}{-s+a} = \frac{1}{s-a}$$

So $\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s>a}$

e.g., if $a=5$, $\mathcal{L}\{e^{5t}\} = \frac{1}{s-5}, \quad s>5$

Ex 5. Let $f(t) = \sin(bt)$. Compute $\mathcal{L}\{\sin(bt)\}$.

By Euler's Formula, $\sin bt = \dfrac{e^{ibt} - e^{-ibt}}{2i}$

$$\mathcal{L}\{\sin bt\} = \int_0^\infty e^{-st}\left(\frac{e^{ibt} - e^{-ibt}}{2i}\right) dt$$

$$= \frac{1}{2i} \lim_{A\to\infty} \int_0^A \left(e^{(-s+ib)t} - e^{(-s-ib)t}\right) dt$$

$$= \frac{1}{2i} \lim_{A\to\infty} \left[\frac{e^{(-s+ib)t}}{-s+ib} - \frac{e^{(-s-ib)t}}{-s-ib}\right]_{t=0}^{t=A}$$

$$= \frac{1}{2i} \lim_{A\to\infty} \left[\frac{e^{(-s+ib)A}}{-s+ib} - \frac{e^{(-s-ib)A}}{-s-ib} - \frac{1}{-s+ib} + \frac{1}{-s-ib}\right]$$

By Euler's Formula, $e^{(-s+ib)A} = e^{-sA}(\cos(bA) + i\sin(bA))$ which converges if and only if e^{-sA} converges, so when $-s < 0$, i.e., $s > 0$.

$$= \frac{1}{2i}\left[-\frac{1}{-s+ib} + \frac{1}{-s-ib}\right]$$

$$= \frac{1}{2i}\left[\frac{s+ib}{(s^2+b^2)} + \frac{-s+ib}{(s^2+b^2)}\right]$$

$$= \frac{1}{2i}\left[\frac{2ib}{s^2+b^2}\right] = \frac{b}{s^2+b^2}$$

so $\boxed{\mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2},\quad s > 0}$