Lesson 27

Diff. Eqs with Discontinuous Forcing Functions (6.4)

In this lesson, we use our knowledge of step functions, the Laplace Transform, and partial fractions to solve IVPs which have discontinuous forcing functions (nonhomogeneous part).

Directions for Examples: Solve the IVP

Ex 1. $y''+3y'+2y = \underbrace{\begin{cases} 1, & 0\le t<10 \\ 0, & t\ge 10, \end{cases}}_{f(t)}$; $y(0)=0$, $y'(0)=0$

$f(t) = 1-u_{10}(t)$

$\mathcal{L}\{y''\}+3\mathcal{L}\{y'\}+2\mathcal{L}\{y\} = \mathcal{L}\{1\}-\mathcal{L}\{u_{10}(t)\}$

$s^2Y(s)-sy(0)-y'(0)+3[sY(s)-y(0)]+2Y(s) = \frac{1}{s}-\frac{e^{-10s}}{s}$

$s^2Y(s)+3sY(s)+2Y(s) = \frac{1}{s}-\frac{e^{-10s}}{s}$

$(s^2+3s+2)Y(s) = \frac{1}{s}-\frac{e^{-10s}}{s}$

$Y(s) = \frac{1}{s(s^2+3s+2)} - \frac{e^{-10s}}{s(s^2+3s+2)}$ $\qquad s^2+3s+2 = (s+2)(s+1)$

$\frac{1}{s(s+2)(s+1)} = \frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+1} \Rightarrow 1 = A(s+2)(s+1)+Bs(s+1)+Cs(s+2)$

$s=0 \Rightarrow 2A=1 \Rightarrow A=\frac{1}{2}$, $s=-2 \Rightarrow 2B=1 \Rightarrow B=\frac{1}{2}$, $s=-1 \Rightarrow -C=1 \Rightarrow C=-1$

$Y(s) = \frac{1}{2}\cdot\frac{1}{s}+\frac{1}{2}\cdot\frac{1}{s+2}-\frac{1}{s+1}+e^{-10s}\left(\frac{1}{2}\cdot\frac{1}{s}+\frac{1}{2}\cdot\frac{1}{s+2}-\frac{1}{s+1}\right)$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}(1)+\frac{1}{2}e^{-2t}-e^{-t}+u_{10}(t)\left[\frac{1}{2}g(t-10)+\frac{1}{2}h(t-10)-j(t-10)\right]$

where $g(t)=\mathcal{L}^{-1}\{\frac{1}{s}\}=1$, $h(t)=\mathcal{L}^{-1}\{\frac{1}{s+2}\}=e^{-2t}$,

$j(t)=\mathcal{L}^{-1}\{\frac{1}{s+1}\}=e^{-t}$

$g(t-10)=1$, $h(t-10)=e^{-2t+20}$, $j(t-10)=e^{-t+10}$

$y(t) = \frac{1}{2}+\frac{1}{2}e^{-2t}-e^{-t}-u_{10}(t)\left[\frac{1}{2}+\frac{1}{2}e^{-2t+20}-e^{-t+10}\right]$

Lesson 27

Ex 2. $y'' + y' + \frac{5}{4}y = t - u_{\pi/2}(t)\left(t - \frac{\pi}{2}\right)$; $y(0) = 0$, $y'(0) = 0$

$t - \frac{\pi}{2} = f\left(t - \frac{\pi}{2}\right)$, so $f(t) = t$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + \frac{5}{4}\mathcal{L}\{y\} = \mathcal{L}\{t\} - \mathcal{L}\{u_{\pi/2}(t) f(t - \tfrac{\pi}{2})\}$$

$$s^2 Y(s) - s y(0) - y'(0) + sY(s) - y(0) + \frac{5}{4}Y(s) = \frac{1}{s^2} - \frac{e^{-\pi/2 s}}{s^2}$$

$$\left(s^2 + s + \frac{5}{4}\right)Y(s) = \frac{1}{s^2} - \frac{e^{-\pi/2 s}}{s^2}$$

$$Y(s) = \frac{1}{s^2(s^2+s+\frac{5}{4})} - \frac{e^{-\pi/2 s}}{s^2(s^2+s+\frac{5}{4})}$$

$$\frac{1}{s^2(s^2+s+\frac{5}{4})} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+s+\frac{5}{4}}$$

$$1 = As\left(s^2+s+\frac{5}{4}\right) + B\left(s^2+s+\frac{5}{4}\right) + (Cs+D)s^2$$

$$1 = As^3 + As^2 + \frac{5}{4}As + Bs^2 + Bs + \frac{5}{4}B + Cs^3 + Ds^2$$

$$\begin{cases} A + C = 0 \\ A + B + D = 0 \\ \frac{5}{4}A + B = 0 \\ \frac{5}{4}B = 1 \end{cases} \Rightarrow B = \frac{4}{5},\ A = -\frac{16}{25},\ C = \frac{16}{25},\ D = \frac{4}{25}$$

Also,
$$\frac{\frac{16}{25}s + \frac{4}{25}}{s^2+s+\frac{5}{4}} = \frac{\frac{16}{25}(s+1) - \frac{8}{25}}{(s+1)^2 + \frac{1}{4}}$$

$$= \frac{16}{25}\frac{(s+1)}{(s+1)^2+\frac{1}{4}} - \frac{16}{25}\frac{\frac{1}{2}}{(s+1)^2+\frac{1}{4}}$$

$$Y(s) = \left[-\frac{16}{25}\cdot\frac{1}{s} + \frac{4}{5}\cdot\frac{1}{s^2} + \frac{16}{25}\frac{(s+1)}{(s+1)^2+\frac{1}{4}} - \frac{16}{25}\cdot\frac{\frac{1}{2}}{(s+1)^2+\frac{1}{4}}\right]\left(1 - e^{-\frac{\pi}{2}s}\right)$$

$$\boxed{\begin{aligned} y(t) = {} & -\frac{16}{25} + \frac{4}{5}t + \frac{16}{25}e^{-t/2}\cos(t) - \frac{16}{25}e^{-t/2}\sin(t) \\ & - u_{\pi/2}(t)\left[-\frac{16}{25} + \frac{4}{5}\left(t - \frac{\pi}{2}\right) + \frac{16}{25}e^{-t/2+\frac{\pi}{4}}\cos\left(t - \frac{\pi}{2}\right) - \frac{16}{25}e^{-t/2+\frac{\pi}{4}}\sin\left(t - \frac{\pi}{2}\right)\right] \end{aligned}}$$

Ex 3. $y'' + y = u_{3\pi}(t)$; $y(0) = 1$, $y'(0) = 0$

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{u_{\pi}(t)\}$

$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{e^{-3\pi s}}{s}$

$s^2 Y(s) - s + Y(s) = \frac{e^{-3\pi s}}{s}$

$(s^2+1)\, Y(s) = s + \frac{e^{-3\pi s}}{s}$

$Y(s) = \frac{s}{s^2+1} + \frac{e^{-3\pi s}}{s(s^2+1)}$

$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow 1 = As^2 + A + Bs^2 + Cs$

$\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \Rightarrow A=1,\ B=-1,\ C=0$

$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$

$Y(s) = \frac{s}{s^2+1} + e^{-3\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$

$y(t) = \cos t + u_{3\pi}(t)\left[f(t-3\pi) - g(t-3\pi)\right]$

$f(t) = \mathcal{L}^{-1}\{\frac{1}{s}\} = 1$, $g(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = \cos t$

$f(t-3\pi) = 1$, $g(t-3\pi) = \cos(t-3\pi) = \cos(t-\pi) = -\cos(t)$

$\boxed{y(t) = \cos t + u_{3\pi}(t)\left[1 + \cos t\right]}$

Ex 4. $y'' - y = u_3(t)$ $y(0) = 0,\ y'(0) = 0$

$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{u_3(t)\}$

$s^2 Y(s) - sy(0) - y'(0) - Y(s) = \frac{e^{-3s}}{s}$

$(s^2-1)\, Y(s) = \frac{e^{-3s}}{s}$

$Y(s) = \frac{e^{-3s}}{s(s^2-1)}$

$\frac{1}{s(s^2-1)} = \frac{A}{s} + \frac{Bs+C}{s^2-1}$

$1 = A(s^2-1) + (Bs+C)s = As^2 - A + Bs^2 + Cs$

$\begin{cases} A+B=0 \\ C=0 \\ -A=1 \end{cases} \Rightarrow A=-1,\ B=1,\ C=0$

$Y(s) = e^{-3s}\left(-\frac{1}{s} + \frac{s}{s^2-1}\right)$

$y(t) = u_3(t)\left[-f(t-3) + g(t-3)\right]$

$f(t) = \mathcal{L}^{-1}\{\frac{1}{s}\} = 1,\quad g(t) = \mathcal{L}^{-1}\{\frac{s}{s^2-1}\} = \cosh t$

$$\boxed{y(t) = u_3(t)\left[\cosh(t-3) - 1\right]}$$

Check: $y(t) = \begin{cases} 0, & 0 \le t < 3 \\ \cosh(t-3) - 1, & t \ge 3 \end{cases} = u_3(t)[\cosh(t-3)-1]$

$y'(t) = \begin{cases} 0, & 0 < t < 3 \\ \sinh(t-3), & t > 3 \end{cases}$

$y''(t) = \begin{cases} 0, & 0 < t < 3 \\ \cosh(t-3), & t > 3 \end{cases} = u_3(t)\cosh(t-3)$

$y''(t) - y(t) = u_3 \cosh(t-3) - u_3(t)[\cosh(t-3) - 1]$

$= u_3(t)$ ✓