Lesson 29

The Convolution Integral (6.6)

Sometimes in computing inverse Laplace transforms, we come across $H(s) = F(s)G(s)$. It is tempting to think that $\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)\} \cdot \mathcal{L}^{-1}\{G(s)\}$, but this is false. To help us with this situation, we define the convolution.

Definition. Let $f(t)$ and $g(t)$ be functions defined on $0 \le t < \infty$. Then the convolution of f and g

$$f * g = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

A nice property is that $f * g = g * f$,

so $\int_0^t f(t-\tau)\, g(\tau)\, d\tau = \int_0^t g(t-\tau)\, f(\tau)\, d\tau$

It turns out that convolutions are the sort of "product" we need for splitting up the Laplace transform.

Thm 6.6.1 Suppose $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Let $h(t) = (f*g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$. Then $H(s) = \mathcal{L}\{h(t)\}$ exists and $H(s) = F(s)G(s)$.

The proof is in the textbook.

Ex 1. Compute $\mathcal{L}\{f(t)\}$ where

$$f(t) = \int_0^t \underbrace{e^{(t-\tau)}}_{g(t-\tau)} \underbrace{\sin 3\tau}_{h(\tau)} \, d\tau$$

so $g(t) = e^t$ and $h(t) = \sin(3t)$

so $f(t) = (g * h)(t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \cdot \mathcal{L}\{h(t)\}$$

$$= \frac{1}{s-1} \cdot \frac{3}{s^2+9}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{3}{(s-1)(s^2+9)}}$$

Ex 2. Find $\mathcal{L}^{-1}\{F(s)\}$ in terms of convolution integrals.

$$F(s) = \frac{1}{(s-3)^3(s^2-16)} = \frac{1}{(s-3)^3} \cdot \frac{1}{(s^2-16)}$$

$$= \underbrace{\left(\frac{1}{2} \cdot \frac{2!}{(s-3)^{2+1}}\right)}_{G(s)} \cdot \underbrace{\left(\frac{1}{4} \cdot \frac{4}{s^2-16}\right)}_{H(s)}$$

$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{1}{2} t^2 e^{3t}$ (line 11)

$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{4} \sinh(4t)$

$g(t-\tau) = \frac{1}{2}(t-\tau)^2 e^{3(t-\tau)}$

$h(\tau) = \frac{1}{4} \sinh(4\tau)$

$$\mathcal{L}^{-1}\{F(s)\} = g * h = \int_0^t \frac{1}{2}(t-\tau)^2 e^{3(t-\tau)} \cdot \frac{1}{4}\sinh(4\tau)\, d\tau$$

$$= \boxed{\frac{1}{8} \int_0^t (t-\tau)^2 e^{3(t-\tau)} \sinh(4\tau)\, d\tau}$$

Lesson 29

Ex 3. Solve the IVP in terms of convolution integrals.

$y'' + 3y' + 2y = \cos \alpha t$; $y(0) = 1$, $y'(0) = 0$

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{\cos\alpha t\}$$

$$s^2Y(s) - sy(0) - y'(0) + 6[sY(s) - y(0)] + 9Y(s) = \mathcal{L}\{\cos\alpha t\}$$

$$s^2Y(s) - s + 6sY(s) - 6 + 9Y(s) = \mathcal{L}\{\cos\alpha t\}$$

$$(s^2 + 6s + 9)Y(s) = s + 6 + \mathcal{L}\{\cos\alpha t\}$$

$$Y(s) = \frac{s+6}{s^2+6s+9} + \frac{1}{s^2+6s+9}\mathcal{L}\{\cos\alpha t\}$$

$$s^2 + 6s + 9 = (s+3)^2$$

$$\frac{s+6}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2}$$

$$s + 6 = A(s+3) + B = As + 3A + B$$

$$A = 1, \quad 3A + B = 6 \Rightarrow B = 3$$

$$Y(s) = \frac{1}{s+3} + \frac{3}{(s+3)^2} + \frac{1}{(s+3)^2}\mathcal{L}\{\cos\alpha t\}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{-3t} + 3te^{-3t} + (f * \cos\alpha t)$$

where $f(t) = \mathcal{L}^{-1}\{\frac{1}{(s+3)^2}\} = te^{-3t}$

$$\boxed{y(t) = e^{-3t} + 3te^{-3t} + \int_0^t (t-\tau)e^{-3(t-\tau)}\cos\alpha\tau \, d\tau}$$

Fun extras: Convolutions can help us solve certain integrals.

Ex 4. Compute $\int_0^5 e^{-x} \sin x \, dx$

We would normally use integration by parts. Instead, write as a convolution integral.

$$\int_0^5 e^{-x} \sin x \, dx = e^{-5} \int_0^5 e^{5-x} \sin x \, dx$$

Let $h(t) = \int_0^t e^{t-\tau} \sin \tau \, d\tau$

Then $\int_0^5 e^{-x} \sin x \, dx = e^{-5} \int_0^5 e^{5-x} \sin x \, dx = e^{-5} h(5)$

$$\mathcal{L}\{e^{-5} h(t)\} = e^{-5} \mathcal{L}\{e^{t}\} \mathcal{L}\{\sin t\}$$
$$= e^{-5} \left(\frac{1}{s-1}\right)\left(\frac{1}{s^2+1}\right)$$

$$\frac{1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)(s-1) = As^2 + A + Bs^2 + Cs - Bs - C$$

$$\begin{cases} A + B = 0 \\ -B + C = 0 \\ A - C = 1 \end{cases} \Rightarrow A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = -\frac{1}{2}$$

$$\mathcal{L}\{e^{-5} h(t)\} = \left[\frac{1}{2}\left(\frac{1}{s-1}\right) - \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1}\right] e^{-5}$$

$$e^{-5} h(t) = \left[\frac{1}{2}e^{t} - \frac{1}{2}\cos t - \frac{1}{2}\sin t\right] e^{-5}$$

$$\boxed{\int_0^5 e^{-x} \sin x \, dx = e^{-5} h(5) = e^{-5}\left[\frac{1}{2}e^{5} - \frac{1}{2}\cos(5) - \frac{1}{2}\sin(5)\right]}$$