Lesson 37

Eigenvalues/vectors and Theory for Systems (7.3, 7.4)

Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a set of vectors. A linear combination is of the form $c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$ for scalars $c_1, \ldots, c_n$.

Notice that the zero vector $\vec{0}$ is always a linear combination of any set of vectors (choose $c_1 = \ldots = c_n = 0$).

A set of vectors is linearly independent if this is the only way to write the $\vec{0}$ vector as a linear combination of the vectors. Otherwise, the set is linearly dependent.

Ex 1. Show that the collection $\{\vec{x}^{(1)}, \vec{x}^{(2)}\}$ is linearly independent where $\vec{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Suppose $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} = \vec{0}$

$$\begin{pmatrix} 2c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2c_1 + c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2c_1 + c_2 = 0 \quad \text{and} \quad c_1 - c_2 = 0$$

$$3c_1 = 0 \Rightarrow c_1 = 0$$

$$0 + c_2 = 0 \Rightarrow c_2 = 0$$

We could also form the matrix $\begin{pmatrix} \vec{x}^{(1)} & \vec{x}^{(2)} \end{pmatrix}$ and take the determinant.

$$\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -2 - 1 = -3 \neq 0$$

A set of 2 2-vectors is linearly independent if and only if the determinant $|\vec{x}^{(1)} \; \vec{x}^{(2)}| \neq 0$.

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Ex 2. Show that $\{\vec{x}^{(1)}, \vec{x}^{(2)}\}$ is linearly dependent where $\vec{x}^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{x}^{(2)} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$

$$\begin{vmatrix} 2 & -4 \\ 3 & -6 \end{vmatrix} = -12 + 12 = 0 \quad \text{or}$$

$$c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} = \vec{0}$$

$$\begin{pmatrix} 2c_1 - 4c_2 \\ 3c_1 - 6c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2c_1 - 4c_2 = 0, \quad 3c_1 - 6c_2 = 0$$

$$6c_1 - 12c_2 = 0, \quad -6c_1 + 12c_2 = 0$$

$$0 = 0$$

Choose any value for c_1. Say $c_1 = 2$

$$4 - 4c_2 = 0 \quad \text{so} \quad c_2 = 1$$

Then $2\vec{x}^{(1)} + 1\vec{x}^{(2)} = \vec{0}$

Let A be a matrix. A scalar λ is called an <u>eigenvalue</u> of A and $\vec{x}$ is called an <u>eigenvector</u> of A associated with λ if $\vec{x} \neq \vec{0}$ and $A\vec{x} = \lambda\vec{x}$.

To find the eigenvalues of A, we get a polynomial, called the <u>characteristic polynomial</u> of A, and find its roots. The characteristic polynomial of A is $\det(A - \lambda I)$, where I is the identity matrix

For 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - cb$$

If λ_0 is an eigenvalue of A, we find an associated eigenvector $\vec{x}^{(0)}$ by solving the system $A\vec{x} = \lambda_0 \vec{x}$, or equivalently, $(A - \lambda_0 I)\vec{x} = \vec{0}$

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If $\vec{x}^{(0)}$ is an eigenvector of A associated with λ_0, then for any scalar $c \neq 0$, $c\vec{x}^{(0)}$ is also such an eigenvector, so we have a choice.

Ex 3. Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

Find char poly: $\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 - (1)(1)$
$= 4 + 4\lambda + \lambda^2 - 1$

$$\lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1)$$

$$\lambda_1 = -1, \quad \lambda_2 = -3$$

To find eigenvectors, solve

$\lambda_1 = -1$: $\left(\begin{array}{cc|c} -2-(-1) & 1 & 0 \\ 1 & -2-(-1) & 0 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right)$

$$\begin{aligned} -x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \end{aligned} \Rightarrow \text{get } x_1 = x_2$$

We can choose any value for x_1, say $x_1 = 1$. Then $x_2 = 1$

so $\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -3$: $\left(\begin{array}{cc|c} -2-(-3) & 1 & 0 \\ 1 & -2-(-3) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right)$

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned} \Rightarrow \text{get } -x_1 = x_2$$

Choose any value for x_1, say $x_1 = 1$, Then $x_2 = -1$.

so $\vec{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\boxed{\lambda_1 = -1,\ \vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \lambda_2 = -3,\ \vec{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

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If A is a real-valued matrix (all entries are real numbers) and λ_1 is a non-real complex number eigenvalue of A, then $\lambda_2 = \overline{\lambda_1}$ is also an eigenvalue. Moreover, if $\vec{x}^{(1)}$ is an eigenvector (assoc. λ_1), then $\vec{x}^{(2)} = \overline{\vec{x}^{(1)}}$ (conjugate) is an eigenvector (assoc. λ_2).

Ex 4. Find eigenvalues and eigenvectors of

$$\begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix}$$

Char poly: $\begin{vmatrix} 1-\lambda & 5 \\ -1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)+5$

$$= -1+\lambda-\lambda+\lambda^2+5$$

$$= \lambda^2+4$$

$$\lambda_1 = 2i, \quad \lambda_2 = -2i$$

Eigenvectors: $\left(\begin{array}{cc|c} 1-2i & 5 & 0 \\ -1 & -1-2i & 0 \end{array}\right)$

$$(1-2i)x_1 + 5x_2 = 0$$

$$-x_1 + (-1-2i)x_2 = 0 \Rightarrow -\frac{(1-2i)}{5}x_1 = x_2$$

Choose $x_1 = 5$. Then $x_2 = -1+2i$.

$$\vec{x}^{(1)} = \begin{pmatrix} 5 \\ -1+2i \end{pmatrix}$$

Then $\vec{x}^{(2)} = \overline{\vec{x}^{(1)}} = \begin{pmatrix} \overline{5} \\ \overline{-1+2i} \end{pmatrix} = \begin{pmatrix} 5 \\ -1-2i \end{pmatrix}$

$$\boxed{\lambda_1 = 2i, \ \vec{x}^{(1)} = \begin{pmatrix} 5 \\ -1+2i \end{pmatrix}; \ \lambda_2 = -2i, \ \vec{x}^{(2)} = \begin{pmatrix} 5 \\ -1-2i \end{pmatrix}}$$

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A system of differential equations can be written as $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$.
If $\vec{g}(t) = \vec{0}$, then the system is <u>homogeneous</u>.

<u>Principle of Superposition</u>. If $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are solutions to a homogeneous system as above, then for any scalars c_1 and c_2, $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is also a solution.

"<u>Wronskian</u>". If $\vec{x}' = P(t)\vec{x}$ is a system of 2 linear ODEs and $\{\vec{x}^{(1)}, \vec{x}^{(2)}\}$ is a linearly independent set of solutions (i.e., $|\vec{x}^{(1)}\,\vec{x}^{(2)}| \neq \vec{0}$), then $\{\vec{x}^{(1)}, \vec{x}^{(2)}\}$ is a fundamental set of solutions.
i.e., all solutions to $\vec{x}' = P(t)\vec{x}$ can be written as $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ for some scalars c_1, c_2.

$c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is called the <u>general solution</u> in this case.