Lesson 34

Repeated Eigenvalues (7.8)

Suppose A has an eigenvalue λ of multiplicity 2.
There are two possibilities:

1. $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$ are linearly independent eigenvectors assoc. with λ. Here, $\vec{x} = c_1 \vec{\xi}^{(1)} e^{\lambda t} + c_2 \vec{\xi}^{(2)} e^{\lambda t}$ is the general solution of $\vec{x}' = A\vec{x}$.
   Behavior along the eigenvectors must be the same, so the origin is a node.
2. There is only one linearly independent eigenvector $\vec{\xi}$ assoc. with λ. This case is much trickier.

Taking inspiration from second order equations, we might guess that the general solution should be...

$$\vec{x} = c_1 \vec{\xi} e^{\lambda t} + c_2 \vec{\xi} t e^{\lambda t}$$

Then $\vec{x}' = c_1 \lambda \vec{\xi} e^{\lambda t} + c_2 \lambda \vec{\xi} t e^{\lambda t} + c_2 \vec{\xi} e^{\lambda t}$

Now, $A\vec{x} = c_1 A\vec{\xi} e^{\lambda t} + c_2 A \vec{\xi} t e^{\lambda t}$

$= c_1 \lambda \vec{\xi} e^{\lambda t} + c_2 \lambda \vec{\xi} t e^{\lambda t}$

In this case, $A\vec{x} = \vec{x}'$ if and only if $c_2 \vec{\xi} e^{\lambda t} = \vec{0}$, but that would require $\vec{\xi} = \vec{0}$, which is not possible, since $\vec{\xi}$ is an eigenvector.

Here, we are off by a factor of $c_2 e^{\lambda t}$, so we assume that our solution is of the form

$$\vec{x} = c_1 \vec{\xi} e^{\lambda t} + c_2 \left[\vec{\xi} t e^{\lambda t} + \vec{\eta} e^{\lambda t} \right]$$

for some vector $\vec{\eta}$.

Lesson 34

Then $\vec{x}' = c_1 \lambda \vec{\xi} e^{\lambda t} + c_2 [\lambda \vec{\xi} t e^{\lambda t} + \vec{\xi} e^{\lambda t} + \lambda \vec{\eta} e^{\lambda t}]$

Also $A\vec{x} = c_1 A \vec{\xi} e^{\lambda t} + c_2 [A\vec{\xi} t e^{\lambda t} + A\vec{\eta} e^{\lambda t}]$

$= c_1 \lambda \vec{\xi} e^{\lambda t} + c_2 [\lambda \vec{\xi} t e^{\lambda t} + A \vec{\eta} e^{\lambda t}]$

For this to be true, $A\vec{\eta} e^{\lambda t} = \vec{\xi} e^{\lambda t} + \lambda \vec{\eta} e^{\lambda t}$

or $A\vec{\eta} = \vec{\xi} + \lambda \vec{\eta}$

or $(A - \lambda I)\vec{\eta} = \vec{\xi}$.

Because of this, $\vec{\eta}$ is called the <u>generalized eigenvector</u> of A.

If $\vec{x}' = A\vec{x}$ and A has a repeated eigenvalue λ but only one linearly independent eigenvector $\vec{\xi}$, solve the system $(A - \lambda I)\vec{\eta} = \vec{\xi}$ to find $\vec{\eta}$. Then the general solution is:

$$\vec{x} = c_1 \vec{\xi} e^{\lambda t} + c_2 [\vec{\xi} t e^{\lambda t} + \vec{\eta} e^{\lambda t}]$$

The behavior of the trajectories in the phase plane depend on the sign of λ.

If $\lambda \geq 0$, then the origin is unstable.

If $\lambda < 0$, then the origin is asymptotically stable.

In such a case, the origin is called an <u>improper node</u>.

Trajectories are like twisted lines.

Lesson 34

Ex 1. Find the general solution.

$$\vec{x}' = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}\vec{x}$$

$$\begin{vmatrix} -\frac{3}{2}-\lambda & 1 \\ -\frac{1}{4} & -\frac{1}{2}-\lambda \end{vmatrix} = \left(-\tfrac{3}{2}-\lambda\right)\left(-\tfrac{1}{2}-\lambda\right) - (1)\left(-\tfrac{1}{4}\right)$$

$$= \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$$

$\lambda = -1$

$$\left(\begin{array}{cc|c} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 \end{array}\right): \quad \begin{aligned} -\tfrac{1}{2}x_1 &= -x_2 \\ -\tfrac{1}{4}x_1 &= -\tfrac{1}{2}x_2 \end{aligned} \Rightarrow \tfrac{1}{2}x_1 = x_2 \quad \vec{\xi} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Find generalized eigenvector $\vec{\eta}$.

$$\left(\begin{array}{cc|c} -\frac{1}{2} & 1 & 2 \\ -\frac{1}{4} & \frac{1}{2} & 1 \end{array}\right): \quad -\tfrac{1}{2}x_1 + x_2 = 2 \Rightarrow \begin{aligned} -\tfrac{1}{2}x_1 - 2 &= -x_2 \\ \tfrac{1}{2}x_1 + 2 &= x_2 \end{aligned} \quad \vec{\eta} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

General solution:

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t}\right]}$$

Lesson 34

Ex 2. Find the general solution.

$$\vec{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{vmatrix} = (4-\lambda)(-4-\lambda) - (8)(-2) = \lambda^2 - 16 + 16$$
$$= \lambda^2$$

so $\lambda = 0$

$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ 8 & -4 & 0 \end{array}\right)$: $\begin{array}{l} 4x_1 - 2x_2 = 0 \\ 8x_1 - 4x_2 = 0 \end{array} \Rightarrow 2x_1 = x_2 \quad \vec{\xi} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Find $\vec{\eta}$

$\left(\begin{array}{cc|c} 4 & -2 & 1 \\ 8 & -4 & 2 \end{array}\right)$: $4x_1 - 2x_2 = 1 \Rightarrow 2x_1 - \frac{1}{2} = x_2 \quad \vec{\eta} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}$

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0t} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t e^{0t} + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} e^{0t} \right]$$

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix} \right]}$$

Lesson 34

On your homework tonight, you are asked to solve a system of equations using the Laplace transform.

Ex 3. Use the Laplace transform to solve

$$\begin{cases} x' = 5x - y, & x(0) = 2 \\ y' = 3x + y, & y(0) = -1 \end{cases}$$

Take $\mathcal{L}$ of both equations

$$\begin{cases} sX(s) - x(0) = 5X(s) - Y(s) \\ sY(s) - y(0) = 3X(s) + Y(s) \end{cases} \Rightarrow \begin{cases} sX(s) - 2 = 5X(s) - Y(s) \\ sY(s) + 1 = 3X(s) + Y(s) \end{cases}$$

Solve the first equation for $Y(s)$ and sub into 2nd

$$Y(s) = (5-s)X(s) + 2$$

$$s[(5-s)X(s)+2] + 1 = 3X(s) + (5-s)X(s) + 2$$

$$(5s - s^2)X(s) + 2s + 1 - 3X(s) + (-5+s)X(s) = 2$$

$$(-s^2 + 6s - 8)X(s) = -2s + 1$$

$$X(s) = \frac{2s-1}{s^2-6s+8} = \frac{2s-1}{(s-4)(s-2)} = \frac{A}{s-4} + \frac{B}{s-2}$$

$2s - 1 = A(s-2) + B(s-4)$ get $A = \frac{7}{2}$, $B = -\frac{3}{2}$

$$X(s) = \frac{7}{2} \cdot \frac{1}{s-4} - \frac{3}{2} \cdot \frac{1}{s-2}$$

$$x(t) = \frac{7}{2} e^{4t} - \frac{3}{2} e^{2t}$$

Plug into first equation and solve for y.

$$x' = 14e^{4t} - 3e^{2t}$$

$$14e^{4t} - 3e^{2t} = \frac{35}{2} e^{4t} - \frac{15}{2} e^{2t} - y(t)$$

$$y(t) = \left(\frac{35}{2} - 14\right)e^{4t} + \left(-\frac{15}{2} + 3\right)e^{2t}$$

$$= \frac{7}{2} e^{4t} - \frac{9}{2} e^{2t}$$

$$\boxed{\begin{aligned} x(t) &= \frac{7}{2} e^{4t} - \frac{3}{2} e^{2t} \\ y(t) &= \frac{7}{2} e^{4t} - \frac{9}{2} e^{2t} \end{aligned}}$$