Lesson 4

Separable Equations (Sec 2.2)

If we have a first order differential equation, we can always write it in the form

$$f(x,y) + g(x,y)\frac{dy}{dx} = 0.$$

If f is a function of only x (no y dependence) and g is a function of only y (no x dependence), we get the form

$$f(x) + g(y)\frac{dy}{dx} = 0$$

Such a diff eq is called separable.

We all know the technique to solve this:

$$g(y)\frac{dy}{dx} = -f(x)$$

$$g(y)\,dy = -f(x)\,dx$$

$$\text{so } \int g(y)\,dy = -\int f(x)\,dx + C$$

But this isn't an incredibly valid technique, though it is often a good way to think about it.

We can show that this is valid rigorously by the chain rule. Let $F(x)$ and $G(y)$ be antiderivatives of $f(x)$ and $g(y)$, respectively.

If $F(x) + G(y) = c$ is differentiated with respect to x, we get

$$F'(x) + G'(y)\frac{dy}{dx} \text{ (chain rule)} = 0$$

$$f(x) + g(y)\frac{dy}{dx} = 0$$

This validates the technique.

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Let's try to use the technique in a few examples.

Ex 1. Solve the diff eq $y' + y^2 \sin x = 0$

$$\frac{dy}{dx} = -y^2 \sin x$$

$$\frac{dy}{y^2} = -\sin x \, dx \qquad \text{if } y \neq 0$$

$$\int \frac{dy}{y^2} = \int -\sin x \, dx$$

$$-\frac{1}{y} = \cos x + C$$

$$\frac{1}{y} = -\cos x + C$$

$$\boxed{y = \frac{1}{\cos x + C}}$$

$$\boxed{\text{Also } y = 0}$$

Often times, solutions will be too difficult to find explicitly, so we can leave our solutions implicit.

Ex 2. $\frac{dy}{dx} = \frac{x^2}{1+3y^2}$

$$(1+3y^2)\,dy = x^2\,dx$$

$$\int (1+3y^2)\,dy = \int x^2\,dx$$

$$y + y^3 = \tfrac{1}{3}x^3 + C$$

so leave solution as

$$\boxed{y + y^3 - \frac{x^3}{3} = C}$$

Note: When we have implicit solutions like this, we must be careful about the domain on which the solution is defined.

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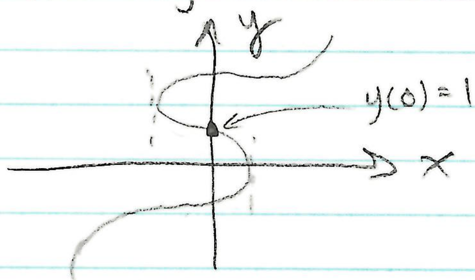
Ex 3. Solve the IVP

$$\frac{dy}{dx} = \frac{1+3x^2}{3y^2-6y}, \quad y(0)=1$$

$$(3y^2-6y)\,dy = (1+3x^2)\,dx$$
$$y^3-3y^2 = x+x^3+C$$
so $y^3-3y^2-x-x^3=C$
$$1-3-0-0=C, \text{ so } C=-2$$
$$y^3-3y^2-x-x^3=-2$$

Where is the solution valid?
Graph it (use Wolfram Alpha or MATLAB)
Looks something like this:



Derivatives do not exist where the function has vertical tangents. We can find these by finding where $\frac{dy}{dx}$ has a denominator equal to 0.

$3y^2-6y=0 \Rightarrow 3y(y-2)=0$ so at $y=0$ and $y=2$.

The initial condition $y(0)=1$ forces us to take the values in between them.

What are the x-values?

When $y=0$, $-x-x^3=-2$. we solve to get $x=1$.
When $y=2$, $-x-x^3=2$. we solve to get $x=-1$.

$$\boxed{\text{So the solution is } y^3-3y^2-x-x^3+2=0 \text{ for } -1<x<1}$$

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Ex 4. Find the solution to the IVP

$$\frac{dy}{dx} = \frac{1-2x}{y}, \quad y(1) = -2$$

$$y\,dy = (1-2x)\,dx$$

$$\frac{y^2}{2} = x - x^2 + C$$

$$\frac{(-2)^2}{2} = 1 - 1 + C, \text{ so } C = 2$$

$$\frac{y^2}{2} = x - x^2 + 2$$

Can solve explicitly:

$$y^2 = 2x - 2x^2 + 4$$

$$y = \pm\sqrt{2x - 2x^2 + 4}$$

We require that when $x=1$, $y=-2$, so we have to choose the negative root.

y is defined when $2x - 2x^2 + 4 > 0$

$$\Rightarrow \quad 2(x^2 - x - 2) < 0$$

$$\Rightarrow \quad x^2 - x + 2 < 0$$

$$\Rightarrow \quad (x-2)(x+1) < 0$$

(y cannot equal 0 - see original diff eq)

So $\boxed{y = -\sqrt{2x - 2x^2 + 4}, \quad -1 < x < 2}$