Lesson 5

Substitutions and Homogeneous Equations

Some nonlinear ~~diff~~ first order diff eqs can be solved by making an appropriate substitution.

Ex 1. Use the substitution $u(x) = y^4$ to solve $4y^3 \frac{dy}{dx} + \frac{y^4}{x} = \frac{1}{x}$, $x > 0$.

If $u = y^4$, then $\frac{du}{dx} = 4y^3 \frac{dy}{dx}$ by the Chain Rule.

so $\frac{du}{dx} + \frac{u}{x} = \frac{1}{x}$, $x > 0$

first order linear

$\mu(x) = \exp\left(\int \frac{1}{x}\, dx\right) = \exp(\ln(x)) = x$, $(x > 0)$

Multiplying both sides by $\mu(x)$

$$\frac{d}{dx}[xu] = 1$$

$$\int \frac{d}{dx}[xu]\, dx = \int 1\, dx$$

$$xu = x + C$$

$$u = 1 + \frac{C}{x}$$

$$y^4 = 1 + \frac{C}{x}$$

$$y = \pm \sqrt[4]{1 + \frac{C}{x}}$$

domain: $1 + \frac{C}{x} \geq 0$

$$1 \geq -\frac{C}{x}$$

$$x \geq -C \quad (x > 0)$$

$$\boxed{y = \pm \sqrt[4]{1 + \frac{C}{x}}, \quad x \geq -C, \quad C < 0.}$$

A particular type of diff eq that can be solved by substitutions are homogeneous equations, which are of the form

$$\frac{dy}{dx} = f(x,y) \text{ where } f(x,y) = g\left(\frac{y}{x}\right)$$

(i.e., f can be expressed as a function of the "variable" $\frac{y}{x}$).

To solve, use the substitution $u(x) = \frac{y}{x}$

Ex 2. Show $\frac{dy}{dx} = \frac{y-4x}{x-y}$ is a homogeneous diff eq., and solve it using the substitution.

$$\frac{dy}{dx} = \frac{y-4x}{x-y}\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \frac{\left(\frac{y}{x}\right)-4}{1-\left(\frac{y}{x}\right)}$$

Let $u(x) = \frac{y}{x}$. Then $xu = y$

So $x\frac{du}{dx} + u = \frac{dy}{dx}$

$$x\frac{du}{dx} + u = \frac{u-4}{1-u}$$

$$x\frac{du}{dx} = \frac{u-4}{1-u} - u$$

$$x\frac{du}{dx} = \frac{u-4}{1-u} - \frac{u(1-u)}{1-u} = \frac{u^2-4}{1-u}$$

$$\frac{1-u}{u^2-4}\,du = \frac{1}{x}\,dx$$

Solve using partial fractions

Lesson 5

$$\frac{1-u}{(u+2)(u-2)}\,du = \frac{1}{x}\,dx$$

$$\frac{1-u}{(u+2)(u-2)} = \frac{A}{u+2} + \frac{B}{u-2}$$

$$1-u = A(u-2) + B(u+2)$$

when $u=-2$, $3=-4A$, so $A = -\frac{3}{4}$

when $u=2$, $-1=4B$, so $B = -\frac{1}{4}$

$$\int \left(-\frac{3}{4}\cdot\frac{1}{u+2} - \frac{1}{4}\cdot\frac{1}{u-2}\right)du = \int \frac{1}{x}\,dx$$

$$-\frac{3}{4}\ln|u+2| - \frac{1}{4}\ln|u-2| = \ln|x| + C$$

$$3\ln|u+2| + \ln|u-2| = -4\ln|x| + C$$

$$\ln|(u+2)^3| + \ln|u-2| = -\ln|x^4| + C$$

$$\ln|(u+2)^3(u-2)x^4| = C$$

$$(u+2)^3(u-2)x^4 = C$$

$$\left(\frac{y}{x}+2\right)^3\left(\frac{y}{x}-2\right)x^4 = C$$

$$\left(\frac{y}{x}+2\right)^3x^3\left(\frac{y}{x}-2\right)x = C$$

$$\boxed{(y+2x)^3(y-2x) = C \quad \text{for } y \neq x}$$

Notice: In the above example, after making the substitution $u(x) = \frac{y}{x}$, the diff eq was separable.

Show this is always true!

Lesson 5

Also, one can see that the direction field of a homogeneous equation is symmetric with respect to the origin.

Why is this true?