Lesson 9

Autonomous Equations and Population Dynamics

A diff eq is <u>autonomous</u> if it is of the form
$\frac{dy}{dt} = f(y)$. (there is no explicit dependence on t)

We looked at these somewhat in lesson 1, but give a more formal treatment here.

We can often obtain a great deal of qualitative information from autonomous equations without even solving them. In this lesson, we do this in the following ways:

Graph $f(y)$ vs. y. The zeros are the <u>equilibrium points</u> or <u>critical points</u>. The y-axis is called the <u>phase line</u> and we can use it to determine how solutions behave. We can then make a qualitatively accurate sketch of the solutions using the phase line.

<u>Ex 1</u>. Draw a qualitatively accurate graph of several solutions to the autonomous diff eq
$$\frac{dy}{dt} = (y-2)(y-5)$$

Notice that $f(y) = (y-2)(y-5)$ is a parabola we start by graphing $f(y)$ vs. y.

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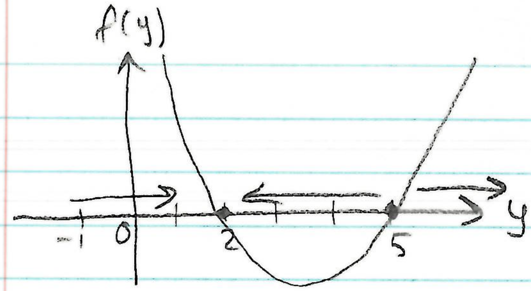
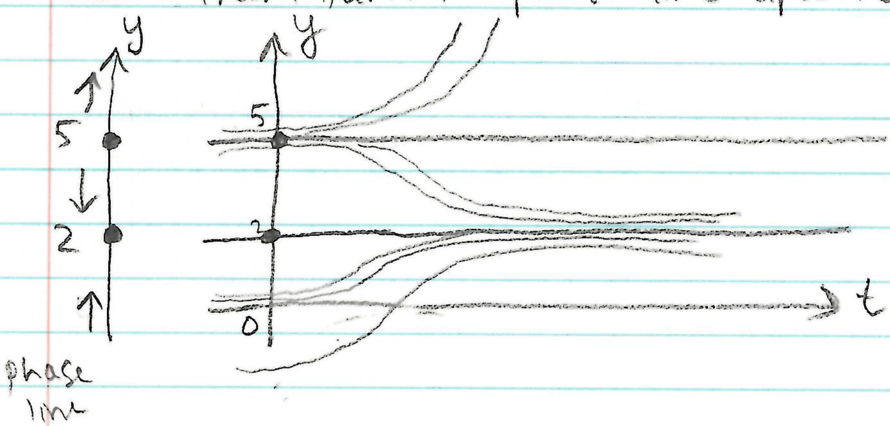


phase line

When $f(y)$ is above the phase line, $\frac{dy}{dt} = f(y)$ is positive, so we draw arrows to the right; when $f(y)$ is below the phase line, $\frac{dy}{dt} = f(y)$ is negative, so we draw arrows to the left.

In this example, $f(2) = 0$ and $f(5) = 0$. These are critical points. (equilibrium solutions).

We then turn the phase line upward:



and draw in the equilibrium solutions on the yt-plane.

Next, we draw in qualitatively accurate solutions based on the arrows of the phase line.

Near critical points, $f(y) = \frac{dy}{dt}$ is close to zero, so solutions are nearly horizontal.

(By Thm 2.4.2, no solutions intersect!)

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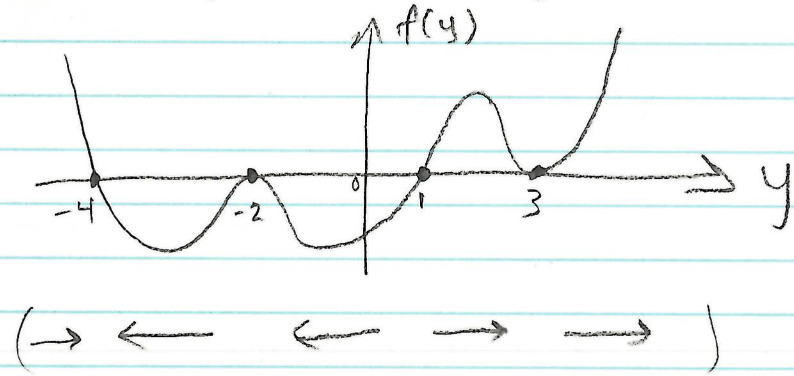
We can classify critical points/equilibrium solutions depending on how other solutions nearby behave.

An equilibrium solution is <u>asymptotically stable</u> if all nearby solutions tend toward the equilibrium (from both sides), is <u>unstable</u> if all nearby solutions tend away from the equilibrium (from both sides), and is <u>semistable</u> if solutions on one side tend toward the equilibrium but solutions on the other side tend away from it.

In Ex 1, $y=2$ is asymptotically stable, and $y=5$ is unstable.

<u>Ex 2</u>. Consider the $f(y)$ vs. y graph given below. Find all equilibrium solutions to $\frac{dy}{dt} = f(y)$ and classify them.



Equilibriums:

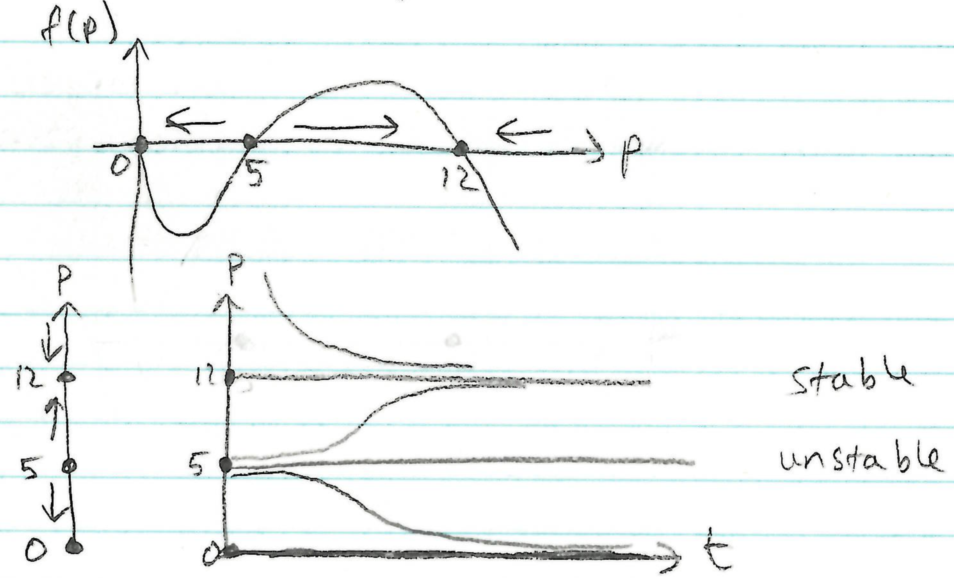
$y=-4$ asymptotically stable
$y=-2$ semistable
$y=1$ unstable
$y=3$ semistable

Autonomous equations are often a good way to model population dynamics, spread of disease, and chemical reactions.

Ex 3. Suppose the population $p(t)$ of a species is given by the autonomous IVP

$$\frac{dp}{dt} = p^2(5-p)(p-12), \quad p(0) = p_0$$

where population is measured in thousands. Determine how the initial population p_0 affects the species in the long run.



If the initial population is less than 5,000, the species will go extinct. If initial population is between 5,000 and 12,000, the population will thrive and stabilize at about 12,000. If initial population is above 12,000, the species will see a decline but stabilize at about 12,000.

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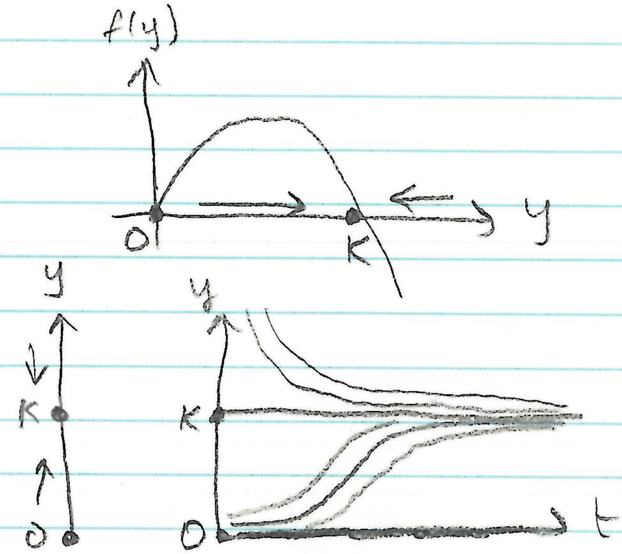
In a population setting, such as in example 3, $p = 5$ (the unstable equilibrium) would be called the threshold (since it determines whether the species thrives or goes extinct). $p = 12$ (the stable equilibrium) would be called the carrying capacity, since any population above it will die out (lack of food, space, etc.) until it once again reaches carrying capacity.

The equation $\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$ is called the logistic equation. It is often used to model population dynamics. You will see another model (the Gompertz equation) on the homework. r and K are constants.

Ex 4. Show that the logistic equation has one stable equilibrium solution, $y = K$.

First notice that $f(y) = r\left(1 - \frac{y}{K}\right)y$ is a parabola with roots $y = 0$ and $y = K$ (opening downward)



The book solves this diff eq and shows what a logistic function looks like algebraically.