MA 265, Lesson 1 (1.1, 1.2)

Systems of Linear Equations (1.1)

What is a linear equation?

In 2-dimensional space, the equation of a line is of the form: $ax+by=c$

In 3-dimensional space: $ax+by+cz=d$

Generalize to n-dimensional space:

$$a_1x_1+a_2x_2+\dots+a_nx_n=b$$

This is a linear equation in n unknowns.

A system of m linear equations in n unknowns is a collection of m linear equations each containing the same n variables.
A solution satisfies all of the equations at the same time.
A system is consistent if it has at least one solution and is inconsistent if it has no solutions.

Ex 1. Determine if the linear system is consistent or inconsistent.

(a)
$$2x_1+3x_2=-1$$
$$4x_1-x_2=-9$$

Use a technique called "elimination".
multiply second equation by 3.

$$2x_1+3x_2=-1$$
$$12x_1-3x_2=-27$$
$$14x_1=-28$$
$$x_1=-2$$

$$2(-2)+3x_2=-1$$
$$-4+3x_2=-1$$
$$3x_2=3$$
$$x_2=1$$

Solution: $x_1=-2$, $x_2=1$ so **consistent**

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(b) $x + 2y = 3$
$-2x - 4y = -7$

multiply top equation by 2

$$2x + 4y = 6$$
$$-2x - 4y = -7$$
$$0 + 0 = -1$$

$0 = -1$ makes no sense!

No solution; so **inconsistent**

(c) $3x_1 + 2x_2 - x_3 = 0$
$3x_1 + x_2 + x_3 = 0$

$6x_1 + 3x_2 = 0$

All constant terms are 0 (called homogeneous solution)

Only one equation... solve for x_2.

$$3x_2 = -6x_1$$
$$x_2 = -2x_1$$

Plug back in to second equation

$$3x_1 - 2x_1 + x_3 = 0$$
$$x_1 + x_3 = 0$$
$$x_3 = -x_1$$

If x_1 = any real number r

$x_1 = r$, $x_2 = -2r$, $x_3 = -r$ is a solution!

consistent

Notice: if $r = 0$, we have $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ as a solution.

Letting every variable = 0 is always a solution to any homogeneous solution and is called the <u>trivial solution</u>

If a homogeneous system has a solution in which not every variable is 0, it is called a <u>nontrivial solution</u>.

Only three possibilities: no solution, a unique solution (exactly one), or infinitely many solutions.

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Ex 2. Find values for s and t so the system is (a) consistent and (b) inconsistent.

$$\begin{aligned} 2x + 2y &= s \\ -x - y &= t \end{aligned} \quad \rightarrow \quad \begin{aligned} 2x + 2y &= s \\ -2x - 2y &= 2t \\ \hline 0 &= s + 2t \end{aligned}$$

$$s = -2t$$

If $\boxed{t = 1,\ s = -2}$

Inconsistent if $\boxed{t = 1, s = 1}$ since we get $0 = 3$

Matrices (1.2)

An $m \times n$ matrix A is a rectangular array of real numbers arranged in m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

← i^{th} row

↑ j^{th} column

The i,j^{th} entry of A is a_{ij}.

If $m = n$, we call A a square matrix and $a_{11}, a_{22}, \ldots, a_{nn}$ are the diagonal entries of A.

An $n \times 1$ matrix is called an n-vector.
Ex: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is a 2-vector.

Sometimes we write a matrix as $A = [a_{ij}]$.

Matrices $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if $a_{ij} = b_{ij}$ for every i and every j (every corresponding entry is equal).

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Matrix Addition: If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then $A+B = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$. If A and B are different sizes, A+B is not defined.

Ex 3. Find A+B, if possible.

(a) $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 4 \\ 2 & -2 \end{bmatrix}$

$A+B = \begin{bmatrix} 2+4 & 0+4 \\ -1+2 & 3+(-2) \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 1 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 6 \\ 4 & 2 & 1 \end{bmatrix}$

A and B are different sizes, so A+B is not defined. (A is 2×2, B is 2×3)

Scalar Multiplication: If $A = [a_{ij}]$ is an $m \times n$ matrix and r is a real number, then $rA = [b_{ij}]$ where $b_{ij} = ra_{ij}$ (multiply every entry of A by r).

Ex 4. Let $A = \begin{bmatrix} 3 & 0 & 4 \\ -2 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

(a) Find $2A = 2\begin{bmatrix} 3 & 0 & 4 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} (2)(3) & (2)(0) & (2)(4) \\ (2)(-2) & (2)(2) & (2)(0) \end{bmatrix} = \begin{bmatrix} 6 & 0 & 8 \\ -4 & 4 & 0 \end{bmatrix}$

(b) Find A − B

$A - B = A + (-1)B$

↑ scalar multiplication

$= \begin{bmatrix} 3 & 0 & 4 \\ -2 & 2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3-1 & 0-1 & 4+0 \\ -2+1 & 2+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 4 \\ -1 & 2 & 0 \end{bmatrix}$

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Let $A_1, A_2, \ldots, A_k$ be $m \times n$ matrices. We call B a <u>linear combination</u> of $A_1, A_2, \ldots, A_k$ if $B = c_1A_1 + c_2A_2 + \ldots + c_kA_k$ for some numbers $c_1, \ldots, c_k$ (called <u>coefficien</u>

<u>Ex 5</u> If possible, write $\begin{bmatrix} 5 & 0 \\ -2 & 10 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$$\begin{bmatrix} 5 & 0 \\ -2 & 10 \end{bmatrix} = r\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + s\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} r+2s & 0 \\ 2r & -r+3s \end{bmatrix}$$

$r + 2s = 5$

$0 = 0$

$2r = -2 \to r = -1$

$-r + 3s = 10$ ✓

$-1 + 2s = 5$

$2s = 6$

$s = 3$

So $\begin{bmatrix} 5 & 0 \\ -2 & 10 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + (3)\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Transpose of a matrix: Given an $m \times n$ matrix A, the <u>transpose of A</u> (A^T) is an $n \times m$ matrix $A^T = [c_{ij}]$ where $c_{ij} = a_{ji}$

<u>Ex 6</u>. Compute A^T:

(a) $A = \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$ $a_{11} = 4$, $a_{12} = 3$, $a_{21} = 0$, $a_{22} = 2$

$c_{11} = a_{11} = 4$, $c_{12} = a_{21} = 0$, $c_{21} = a_{12} = 3$, $c_{22} = a_{22} = 2$

So $A^T = \begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}$ $a_{11} = 2$, $a_{12} = 3$, $a_{21} = 4$, $a_{22} = -1$, $a_{31} = 0$, $a_{32} = 2$

$c_{11} = a_{11} = 2$, $c_{12} = a_{21} = 4$, $c_{13} = a_{31} =$

$c_{21} = a_{12} = 3$, $c_{22} = a_{22} = -1$, $c_{23} = c_{32} =$

$$A^T = \begin{bmatrix} 2 & 4 & 0 \\ 3 & -1 & 2 \end{bmatrix}$$

Notice: The i^{th} row of A^T is the i^{th} column of A (and vice versa)