Lesson 13

Repeated Roots and Reduction of Order (3.4)

When the characteristic polynomial ar^2+br+c has a repeated real root, we run into some problems. We know that $y(t) = e^{rt}$ is a solution, but what about the other solution?

Jean d'Alembert (1717-1783) developed a technique for this situation. It is called the method of reduction of order. As the name suggests, using the solution, we use a clever trick to reduce the order of the differential equation to find another solution, which together form a fundamental set.

Suppose $ay''+by'+cy=0$ has a characteristic polynomial with repeated root r. Notice $r = -\frac{b}{2a}$.

So we have $y_1 = e^{rt}$ as a solution. We assume $y = v(t)y_1 = v(t)e^{rt}$ is another solution, for some function $v(t)$.

Then $y' = re^{rt}v + e^{rt}v'$ and

$$y'' = re^{rt}v' + r^2e^{rt}v + e^{rt}v'' + re^{rt}v'$$
$$= e^{rt}(v'' + 2rv' + r^2v)$$

Then $ae^{rt}(v''+2rv'+r^2v) + be^{rt}(v'+rv)$
$+ce^{rt}v = 0$

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Simplifying, we get...

$$\underbrace{e^{rt}}_{\text{never zero}}(av'' + 2arv' + ar^2v + bv' + brv + cv) = 0$$

$$av'' + \underbrace{(2ar+b)}_{=0 \text{ since } r = -\frac{b}{2a}}v' + \underbrace{(ar^2+br+c)}_{=0 \text{ since } r \text{ is a root}}v = 0$$

$av'' = 0$, or equivalently, $v'' = 0$

Integrating, we get $v' = c_1$, then again, $v = c_1 t + c_2$.

Since $y_2 = ve^{rt}$, we have $y = c_1 te^{rt} + c_2 e^{rt}$, so te^{rt} is a solution as well.

We check that $\{e^{rt}, te^{rt}\}$ is a fundamental set.

$$\begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & rte^{rt} + e^{rt} \end{vmatrix} = rte^{2rt} + e^{2rt} - rte^{2rt}$$

$$= e^{2rt} \neq 0$$

Thus, if the characteristic polynomial has a repeated root r, the general solution is

$$y(t) = c_1 e^{rt} + c_2 te^{rt}$$

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Ex 1. Find the general solution to

$$y'' - 4y' + 4y = 0$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = 2$$

so $\boxed{y(t) = c_1 e^{2t} + c_2 t e^{2t}}$

Ex 2. Find the general solution to

$$2y'' + 12y' + 18y = 0$$

$$2r^2 + 12r + 18 = 0$$

$$r = \frac{-12 \pm \sqrt{144 - 144}}{4} = -3$$

so $\boxed{y(t) = c_1 e^{-3t} + c_2 t e^{-3t}}$

The method of reduction of order works for other situations too!

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Method of Reduction of Order

Given a second order linear equation and one solution $y_1(t)$, assume $y(t) = v(t)\,y_1(t)$.
Then calculate y' and y''.
Plug these into the diff eq.
After doing algebra, you will end up with a first order equation in v'. Solve this for v'.
Then integrate to get v. $v(t)\,y_1(t)$ is the other solution.

Ex 3. Find another solution to
$$t^2y'' + 2ty' - 2y = 0,\ t>0;\ y_1(t) = t.$$

Assume $y(t) = v(t)\,y_1(t) = v(t)\,t$

Then $y' = v(t) + tv'(t)$

$y'' = v'(t) + tv''(t) + v'(t) = tv''(t) + 2v'(t)$

$$t^2(tv'' + 2v') + 2t(v + tv') - 2(tv) = 0$$
$$t^3v'' + 2t^2v' + \underline{2tv} + 2t^2v' - \underline{2tv} = 0$$

$$t^3v'' + 4t^2v' = 0$$

This is first order in v'. Let $u = v'$, Then $u' = v''$.

$$t^3u' + 4t^2u = 0$$
$$u' + \frac{4}{t}u = 0$$
$$\mu(t) = \exp\left(\int \frac{4}{t}\,dt\right) = e^{4\ln t} = t^4$$
$$\frac{d}{dt}[t^4u] = 0$$
$$t^4u = k_1$$
$$v' = u = k_1t^{-4}$$

So $v = -\frac{k_1}{3}t^{-3} + k_2$

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Thus $y(t) = v(t) \cdot t = \underbrace{\frac{-k_1}{3}}_{\text{constant}} t^{-2} + \overbrace{k_2 t}^{y_1}$

Hence, the other solution is just t^{-2}.

Check the Wronskian to be sure:

$$\begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0$$

So it is legitimately a different solution.

$\boxed{y_2(t) = t^{-2}}$ so the general solution is

$\boxed{y(t) = c_1 t + c_2 t^{-2}}$

One can show in general that this procedure always works in forming a first order equation in v'.
The book shows this too.