Lesson 29

Complex Eigenvalues (7.6)

If A is a real 2×2 matrix and $\vec{u}+i\vec{v}$ is a solution to $\vec{x}'=A\vec{x}$ ($\vec{u}$ and $\vec{v}$ are real 2-vectors) then both $\vec{u}$ and $\vec{v}$ are solutions as well.
($\vec{u}'+i\vec{v}' = A\vec{u}+iA\vec{v}$, so $\vec{u}'=A\vec{u}$ and $\vec{v}'=A\vec{v}$)

That being said, we can express our solution in terms of real values given one eigenvalue λ and one eigenvector $\vec{\xi}$ (when we have complex conjugate eigenvalues). To be clear this is <u>only true</u> when A is a <u>real</u> matrix (all entries are real numbers).

To find the solution of $\vec{x}'=A\vec{x}$ when A is real and has complex eigenvalues $\lambda \pm i\mu$, find an eigenvector $\vec{\xi}$ associated with $\lambda+i\mu$.

Then one solution is of the form

$$\vec{\xi}e^{(\lambda+i\mu)t} = \vec{\xi}\left(e^{\lambda t}[\cos(\mu t)+i\sin(\mu t)]\right)$$

Distribute into the vector $\vec{\xi}$.
Separate into two vectors $\vec{u}+i\vec{v}$.
Then the solutions are of the form

$$c_1 e^{\lambda t}\vec{u} + c_2 e^{\lambda t}\vec{v}.$$

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Ex 1. Complex eigenvalues with positive real part.

$$\vec{x}\,' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \vec{x}$$

Find eigenvalues:

$$\begin{vmatrix} 2-\lambda & -\frac{5}{2} \\ \frac{9}{5} & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - \left(\tfrac{9}{5}\right)\left(-\tfrac{5}{2}\right) = \lambda^2 - \lambda - 2 + \tfrac{9}{2}$$

$$\lambda^2 - \lambda + \tfrac{5}{2}$$

$$\lambda = \frac{1 \pm \sqrt{1-10}}{2} = \frac{1 \pm 3i}{2} = \frac{1}{2} \pm \frac{3}{2} i$$

$$\left(\begin{array}{cc|c} \frac{3}{2} - \frac{3}{2} i & -\frac{5}{2} & 0 \\ \frac{9}{5} & -\frac{3}{2} - \frac{3}{2} i & 0 \end{array}\right) \qquad \begin{array}{l} \left(\frac{3}{2} - \frac{3}{2} i\right) x_1 = \frac{5}{2} x_2 \\ \left(\frac{3}{5} - \frac{3}{5} i\right) x_1 = x_2 \end{array} \qquad \vec{\xi} = \begin{pmatrix} 5 \\ 3-3i \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 3-3i \end{pmatrix} e^{t/2} \left(\cos\left(\tfrac{3}{2}t\right) + i \sin\left(\tfrac{3}{2}t\right) \right)$$

$$= e^{t/2} \begin{pmatrix} 5\cos\left(\frac{3}{2}t\right) + 5i\sin\left(\frac{3}{2}t\right) \\ 3\cos\left(\frac{3}{2}t\right) + 3i\sin\left(\frac{3}{2}t\right) - 3i\cos\left(\frac{3}{2}t\right) + 3\sin\left(\frac{3}{2}t\right) \end{pmatrix}$$

$$= e^{t/2} \begin{pmatrix} 5\cos\left(\frac{3}{2}t\right) \\ 3\cos\left(\frac{3}{2}t\right) + 3\sin\left(\frac{3}{2}t\right) \end{pmatrix} + e^{t/2}\, i \begin{pmatrix} 5\sin\left(\frac{3}{2}t\right) \\ 3\sin\left(\frac{3}{2}t\right) - 3\cos\left(\frac{3}{2}t\right) \end{pmatrix}$$

so

$$\boxed{\vec{x}(t) = c_1 e^{t/2} \begin{pmatrix} 5\cos\left(\frac{3}{2}t\right) \\ 3\cos\left(\frac{3}{2}t\right) + 3\sin\left(\frac{3}{2}t\right) \end{pmatrix} + c_2 e^{t/2} \begin{pmatrix} 5\sin\left(\frac{3}{2}t\right) \\ 3\sin\left(\frac{3}{2}t\right) - 3\cos\left(\frac{3}{2}t\right) \end{pmatrix}}$$

Use pplane8 to graph.

Notice trajectories are all spirals, so $\vec{0}$ is called a <u>spiral point</u>.

As $t \to \infty$, $\vec{x}(t)$ diverges from $\vec{0}$, so $\vec{0}$ is an <u>unstable spiral point</u>.

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Ex 2. Complex eigenvalues with negative real part

$\vec{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \vec{x}$

$\begin{vmatrix} 1-\lambda & -1 \\ 5 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) - (5)(-1) = \lambda^2 + 2\lambda - 3 + 5$

$\lambda^2 + 2\lambda + 2$

$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

$\left(\begin{array}{cc|c} 2-i & -1 & 0 \\ 5 & -2-i & 0 \end{array}\right) : \quad (2-i)x_1 = x_2 \qquad \vec{\xi} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

$e^{-t} \begin{pmatrix} 1 \\ 2-i \end{pmatrix} [\cos(t) + i\sin(t)]$

$= e^{-t} \begin{pmatrix} \cos(t) + i\sin(t) \\ 2\cos(t) + 2i\sin(t) - i\cos(t) + \sin(t) \end{pmatrix}$

$= e^{-t} \left[\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} \right]$

$$\boxed{\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}}$$

Plot with pplane8.

As $t \to \infty$, trajectories converge to $\vec{0}$.
Trajectories are spirals toward $\vec{0}$,
So $\vec{0}$ is an <u>asymptotically stable spiral point</u>.

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Ex 3. Purely imaginary eigenvalues

$\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$

$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - (1)(-5) = \lambda^2 - 4 + 5$

$\lambda^2 + 1$

$\lambda = \pm i$

$\left(\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right)$: $(2-i)x_1 = 5x_2$ $\quad \vec{\xi} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$

$\frac{2-i}{5} x_1 = x_2$

$\begin{pmatrix} 5 \\ 2-i \end{pmatrix} [\cos(t) + i\sin(t)] = \begin{pmatrix} 5\cos t + 5i\sin t \\ 2\cos t + 2i\sin t - i\cos t + \sin t \end{pmatrix}$

$= \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + i \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$

$$\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}}$$

Plot with pplane8

All trajectories are elliptical, so $\vec{0}$ is called a <u>center</u>.

As $t \to \infty$, the trajectories neither converge nor diverge, so we say $\vec{0}$ is <u>stable</u> (but not asymptotically stable).

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Critical values for phase portraits:

We obtain critical values when the behavior changes from one form to another.

- Changes from complex to real or vice versa (nodes or saddle points to spiral or center points)
- Changes from complex to purely imaginary or vice versa (spiral points to center points)
- Eigenvalue $= 0$. (changes from one type to another)

Ex 4. Find the critical values of α where the qualitative nature of the phase portrait changes.

$$\vec{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -\lambda & -5 \\ 1 & \alpha-\lambda \end{vmatrix} = (-\lambda)(\alpha-\lambda) - (1)(-5) = \lambda^2 - \alpha\lambda + 5$$

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 - 20}}{2}$$

Possibilities: Changes from complex to real or vice versa

Occurs when $\alpha^2 - 20 = 0$ i.e., when $\boxed{\alpha = \pm\sqrt{20}}$

Eigenvalues are complex when $-\sqrt{20} < \alpha < \sqrt{20}$

When $\lambda = 0$:

$$\alpha - \sqrt{\alpha^2 - 20} = 0$$
$$\alpha = \sqrt{\alpha^2 - 20}$$
$$\alpha^2 = \alpha^2 - 20$$
$$0 = -20$$

$\boxed{\text{This type never occurs}}$

Purely imaginary eigen values occur when $\boxed{\alpha = 0.}$

So critical values are

$$\boxed{\alpha = -\sqrt{20} \text{ or } 0 \text{ or } \sqrt{20}}$$