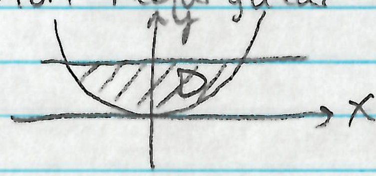
MA 261 - Lesson 12

Double Integrals over Arbitrary Regions and by Polar Coordinates

15.2 Double Integrals over Arbitrary Regions

Suppose you want to take a double integral over a non-rectangular region of the plane, e.g.,



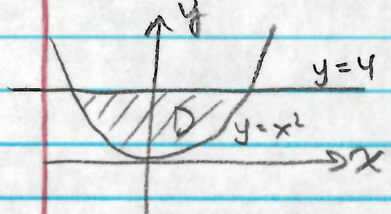
By the same argument used in Fubini's Theorem, we can do it by using iterated integrals as $dx\,dy$ or $dy\,dx$, but now it's a bit more complicated.

For the inner integral, your bounds of integration are now curves instead of just numbers.

For a $dx\,dy$ integral, the inner integral has its lower bound being the curve $x = f(y)$ bounding D on the left and upper bound being the curve $x = g(y)$ bounding D on the right. The outer integral has lower bound being the minimum y-value attained in D and upper bound being the maximum y-value attained in D.

Similarly, for a $dy\,dx$ integral, the inner integral has lower bound the curve $y = f(x)$ bounding D below and upper bound the curve $y = g(x)$ bounding D above. The outer integral has lower bound being the minimum x-value attained in D and upper bound being the maximum x-value attained in D.

Ex 1. Set up two double integrals representing the volume of the surface $z = f(x,y)$ over the region D bounded by $y = x^2$ and $y = 4$.



dy dx

D bounded below by curve $y = x^2$

above by curve $y = 4$

D has minimum x-value -2

maximum x-value 2

$$\boxed{\int_{-2}^{2}\int_{x^2}^{4} f(x,y)\,dy\,dx}$$

dx dy

D bounded on the left by curve $x = -\sqrt{y}$

right by curve $x = \sqrt{y}$

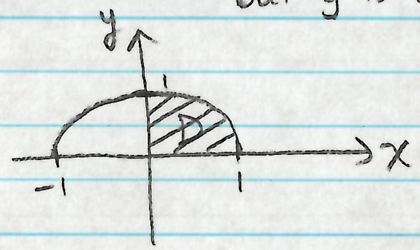
D has minimum y-value 0

maximum y-value 4

$$\boxed{\int_{0}^{4}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy}$$

Ex 2. Evaluate the double integral $\iint_D xy \, dA$ where D is enclosed by the quarter circle $y = \sqrt{1-x^2}$, $x \geq 0$, and the axes

If $y = \sqrt{1-x^2}$, then $y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$
but y is also positive



dy dx

D bounded below by curve $y = 0$
above by curve $y = \sqrt{1-x^2}$
D has minimum x-value 0
max x-value 1

$$\int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx$$

$$= \int_0^1 \frac{1}{2} x y^2 \Big|_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_0^1 \left(\frac{1}{2} x \left(\sqrt{1-x^2}\right)^2 - \frac{1}{2} x (0)^2 \right) dx$$

$$= \int_0^1 \frac{1}{2} (x - x^3) \, dx$$

$$= \frac{1}{2} \left(\frac{1}{2} x^2 - \frac{1}{4} x^4 \right) \Big|_0^1 = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} - 0 + 0 \right) = \boxed{\frac{1}{8}}$$

Could also do dxdy

$$\int_0^1 \int_0^{\sqrt{1-y^2}} xy \, dx \, dy = \frac{1}{8}$$

Sometimes, an iterated integral is impossible or difficult to do in its current order of integration, and you should switch.

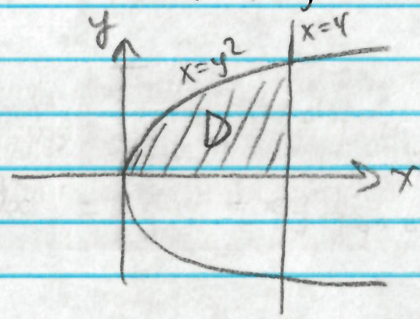
Ex 3. Evaluate $\int_0^2 \int_{y^2}^4 y e^{x^2} \, dx \, dy$

Impossible to do without power series! Switch order.

dx dy

D is bounded on the left by curve $x = y^2$
right by curve $x = 4$

D has min y-value 0
max y-value 2



dy dx

D bounded below by curve $y = 0$
above by curve $y = \sqrt{x}$

D has min x-value 0
max x-value 4

$$\int_0^4 \int_0^{\sqrt{x}} y e^{x^2} \, dy \, dx$$

$$= \int_0^4 \tfrac{1}{2} y^2 e^{x^2} \Big|_{y=0}^{y=\sqrt{x}} dx = \int_0^4 \left(\tfrac{1}{2}(\sqrt{x})^2 e^{x^2} - \tfrac{1}{2}(0)^2 e^{x^2}\right) dx$$

$$= \int_0^4 \tfrac{1}{2} x e^{x^2} dx \qquad u = x^2,\ du = 2x\,dx$$

$$= \tfrac{1}{4} \int_0^{16} e^u \, du$$

$$= \tfrac{1}{4} e^u \Big|_{u=0}^{u=16} = \tfrac{1}{4}(e^{16} - e^0) = \boxed{\tfrac{1}{4} e^{16} - \tfrac{1}{4}}$$

Average Value of $f(x,y)$ over a region D

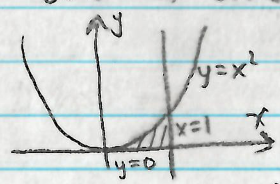
For a single variable function $y=f(x)$, the average value over an interval $[a,b]$ is

$$\frac{1}{\text{Length}([a,b])}\int_a^b f(x)\,dx$$

For $z=f(x,y)$ over a region D, the average value is

$$\frac{1}{\text{Area}(D)}\iint_D f(x,y)\,dA$$

Ex. 4. Find the average value of $f(x,y)=x\sin y$ over D, where D is enclosed by the curves $y=0, y=x^2, x=1$.



$$\int_0^1\int_0^{x^2} x\sin y\,dy\,dx$$

$$=\int_0^1 -x\cos y\Big|_{y=0}^{y=x^2}dx = -\int_0^1 (x\cos x^2 - x)\,dx$$

$$=-\int_0^1 x\cos x^2\,dx + \int_0^1 x\,dx$$

$u=x^2,\ du=2x\,dx$

$$= -\int_0^1 \tfrac{1}{2}\cos u\,du + \tfrac{1}{2}x^2\Big|_0^1$$

$$= -\tfrac{1}{2}\sin u\Big|_{u=0}^{u=1} + \left(\tfrac{1}{2}-0\right)$$

$$= -\tfrac{1}{2}\sin(1)+0+\tfrac{1}{2}-0 = \tfrac{1}{2}(1-\sin(1))$$

$$\text{Area}(D) = \int_0^1 x^2\,dx = \tfrac{1}{3}x^3\Big|_0^1 = \tfrac{1}{3}$$

$$\text{Average value} = \frac{1}{(\frac{1}{3})}\iint_D x\sin y\,dA = 3\left(\tfrac{1}{2}(1-\sin(1))\right)$$

$$= \boxed{\tfrac{3}{2}(1-\sin(1))}$$

15.3 Double Integrals by Polar Coordinates

Every point in the xy-plane can be written in polar coordinates (r, θ), where r is the distance of the point from the origin and θ is the it makes from the positive x-axis.

As such, given a point (x,y), we have

$r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ (adjusted to be in the proper quadrant, possibly)

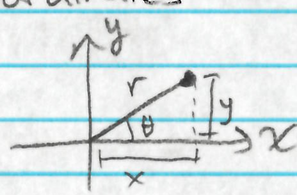
So $(\sqrt{2}, \sqrt{2})$ in rectangular coordinates has

$r = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{2+2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$ (in QI ✓)

so has polar coordinates $\left(2, \frac{\pi}{4}\right)$

Given a point (r, θ) in polar coordinates, we can use geometry to convert back to rectangular coordinates



$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

So if $\left(3, \frac{\pi}{3}\right)$ is polar coordinates, it has

$x = 3\cos\left(\frac{\pi}{3}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$

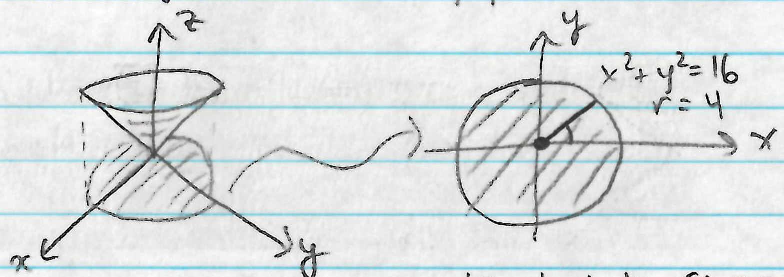
$y = 3\sin\left(\frac{\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$

$\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ as rectangular coordinates

Polar coordinates are good at representing circular regions ($r=4$ is a circle of radius 4), so sometimes it is nice to convert a double integral into polar coordinates using $r^2 = x^2+y^2$, $x = r\cos\theta$, $y = r\sin\theta$.

The differential $dA = dx\,dy$ or $dy\,dx$ can be converted to polar as $r\,dr\,d\theta$ (described in book)

<u>Ex 5</u>. Use polar coordinates to evaluate the volume under the cone $z=\sqrt{x^2+y^2}$ and above the circle $x^2+y^2=16$ in the xy-plane.



$z = \sqrt{x^2+y^2} = r$

notice that to cover the whole circle, r varies between 0 and 4 and θ varies between 0 and 2π

$$\int_0^{2\pi}\int_0^4 \overset{\sqrt{x^2+y^2}=z}{\overset{\|}{r}} \cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^4 r^2\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}r^3\Big|_0^4\,d\theta$$

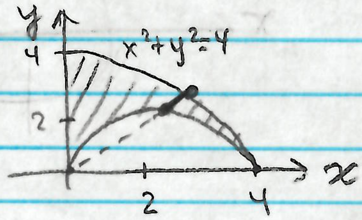
$$= \int_0^{2\pi} \frac{64}{3}\,d\theta$$

$$= \frac{64}{3}\theta\Big|_0^{2\pi}$$

$$= \boxed{\frac{128}{3}\pi}$$

Ex 6. Evaluate $\iint_D y \, dA$, where D is the region in the first quadrant that lies between the circles $x^2+y^2=16$ and $x^2+y^2=4x$

$$x^2+y^2=4x \iff x^2-4x+4+y^2=4 \iff (x-2)^2+y^2=4$$



Notice here, r varies from the lower circle

$(x^2+y^2=4x \iff r^2=4r\cos\theta \iff r=4\cos\theta)$

to the upper circle

$(x^2+y^2=16 \iff r=4)$

and θ varies from 0 to $\frac{\pi}{2}$

Also $y=r\sin\theta$

$$\int_0^{\frac{\pi}{2}}\int_{4\cos\theta}^{4} r\sin\theta \cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_{4\cos\theta}^{4} r^2\sin\theta \,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{3}r^3\sin\theta \Big|_{r=4\cos\theta}^{r=4} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{64}{3}\sin\theta - \frac{64}{3}\cos^3\theta\sin\theta\right) d\theta$$

$$= \frac{64}{3}\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta - \frac{64}{3}\int_0^{\frac{\pi}{2}} \cos^3\theta \sin\theta \, d\theta$$

$u=\cos\theta, \quad du=-\sin\theta \, d\theta$

$$= \frac{64}{3}\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta + \frac{64}{3}\int_1^0 u^3 \, du$$

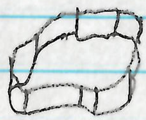
$$= -\frac{64}{3}\cos\theta \Big|_0^{\frac{\pi}{2}} + \frac{64}{3}\cdot\frac{1}{4}u^4 \Big|_1^0$$

$$= \left(0+\frac{64}{3}\right) + \left(0-\frac{\cancel{64}^{16}}{3}\cdot\frac{1}{\cancel{4}}\right)$$

$$= \frac{64}{3} - \frac{16}{3} = \frac{48}{3} = \boxed{16}$$

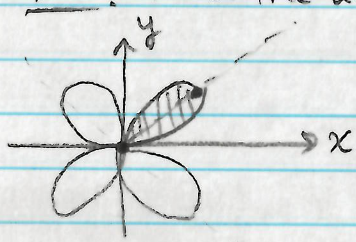
Areas of Polar regions

Double integrals can be used to find areas of regions D: just find the volume under $z=1$ above D (must = area of D)



height is 1, so volume = $1 \cdot$ Area of D = Area of D

Ex 7. Find the area of one loop of the rose $r = \sin 2\theta$



One loop can be obtained by r varying from 0 to $\sin 2\theta$ and θ varying from 0 to $\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} 1 \cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \Big|_0^{\sin 2\theta} d\theta$$

Recall: $\left(\sin^2 t = \frac{1-\cos(2t)}{2} \right)$

$$= \int_0^{\pi/2} \frac{1}{2} \sin^2(2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}\left(1 - \cos(4\theta)\right) d\theta$$

$$= \frac{1}{4}\left(\theta - \frac{1}{4}\sin(4\theta)\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{4}\Big(\frac{\pi}{2} - \underbrace{\frac{1}{4}\sin(2\pi)}_{0}\Big) - \frac{1}{4}\Big(0 - \underbrace{\frac{1}{4}\sin(0)}_{0}\Big)$$

$$= \boxed{\frac{\pi}{8}}$$