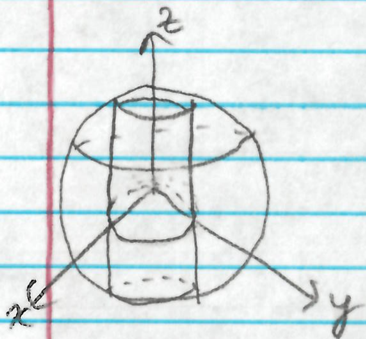
MA 261 – Lesson 13

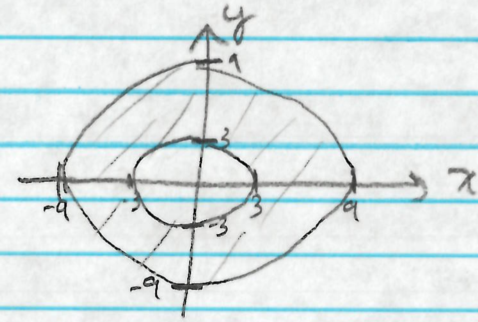
Applications of Double Integrals

15.3 Review on Polar

Ex 1. Set up a double integral in polar coordinates for the volume inside the sphere $x^2+y^2+z^2=81$ and outside the cylinder $x^2+y^2=9$



project onto xy-plane



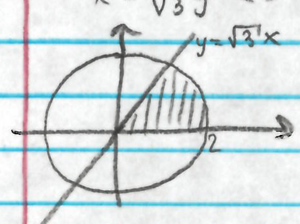
below the surface $z^2=81-x^2-y^2$ or $z=\sqrt{81-x^2-y^2}$
$=\sqrt{81-r^2}$

(top half above the plane, and double volume to get whole thing)

$$\boxed{2\int_0^{2\pi}\int_3^9 \sqrt{81-r^2}\; r\,dr\,d\theta}$$

Ex 2. Convert $\int_0^{\sqrt{3}}\int_{\frac{1}{\sqrt{3}}y}^{\sqrt{4-y^2}} yx^2\,dx\,dy$ into polar.

$x=\frac{1}{\sqrt{3}}y \iff y=\sqrt{3}x$ $\quad x=\sqrt{4-y^2} \iff x^2+y^2=4$, but the right half



Notice that the line intersects the circle when $y=\sqrt{3}$

when $y=\sqrt{3}$, $x=1$ so $\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\theta$

$\theta=\frac{\pi}{3}$

$$\int_0^{\frac{\pi}{3}}\int_0^2 (r\sin\theta)(r\cos\theta)^2\, r\,dr\,d\theta$$

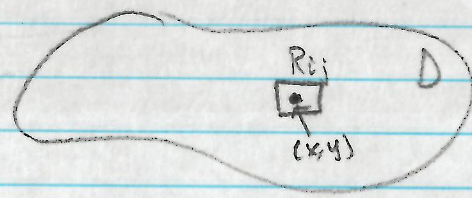
$$=\boxed{\int_0^{\frac{\pi}{3}}\int_0^2 r^4\cos^2\theta\sin\theta\,dr\,d\theta}$$

15.4 Laminas and Centroids

In Calc 2, you found the centroid of a lamina with uniform density using single integrals. Double integrals allow us to find the mass and centroid of a lamina where the density varies throughout the lamina.

Suppose D is a lamina where the density at the point (x,y) in D is $\rho(x,y)$.

Since average density is the ratio of the mass over a region to the area of that region, $\rho(x,y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$ (where A is the area of a region containing (x,y) and m is the mass of that region).



To approximate the mass of D, consider a small rectangle R_{ij} containing (x_{ij}, y_{ij}). The average mass on R_{ij} is approximately $\rho(x_{ij}, y_{ij})\,\Delta A$ where ΔA is the area of R_{ij}, and this approximation gets better as $\Delta A \to 0$.

Thus, an approximation for the mass m of D is $\sum_{i=1}^{k}\sum_{j=1}^{\ell} \rho(x_{ij}, y_{ij})\Delta A$ and this gets better as $(k,\ell) \to (\infty,\infty)$.

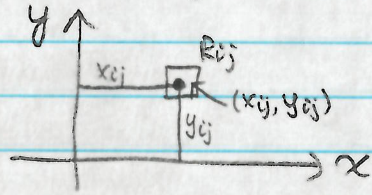
Hence, the mass m of a lamaina D with density $\rho(x,y)$ is

$$\boxed{m = \iint_D \rho(x,y)\, dA}$$

The same reasoning applies to other types of densities, such as electric charge and probability.

Recall from calc 2 that the moment about an axis of a particle with mass m and directed distance d from the axis is md.

Thus the moment about the x-axis is approximately $\rho(x_{ij}, y_{ij})\Delta A \cdot y_{ij}$ for a rectangle R_{ij} containing (x_{ij}, y_{ij}) and similarly the moment about the y-axis is approximately $\rho(x_{ij}, y_{ij})\Delta A \cdot x_{ij}$



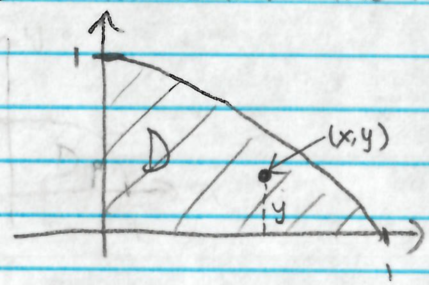
Thus, for the whole lamina D,

$$M_x = \iint_D y\,\rho(x,y)\,dA, \quad M_y = \iint_D x\,\rho(x,y)\,dA$$

The center of mass $(\bar{x}, \bar{y})$ of a lamina D is

$$\boxed{\begin{aligned} \bar{x} &= \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA \\ \bar{y} &= \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA \end{aligned}}$$

Ex 3. A lamina occupies the part of the disk $x^2+y^2 \le 1$ in the first quadrant. Find its center of mass if its density at any point is proportional to its distance from the x-axis.



The distance of a point (x,y) to the x-axis is y.
So $\rho(x,y) = ky$, for some constant k.

$$m = \iint_D ky\, dA = \int_0^{\frac{\pi}{2}}\int_0^1 kr\sin\theta \, r\,dr\,d\theta$$
$$= \int_0^{\pi/2} k\frac{r^3}{3}\sin\theta \Big|_0^1 d\theta = \frac{k}{3}\int_0^{\pi/2} \sin\theta\, d\theta$$
$$= -\frac{k}{3}\cos\theta \Big|_0^{\pi/2} = \frac{k}{3}$$

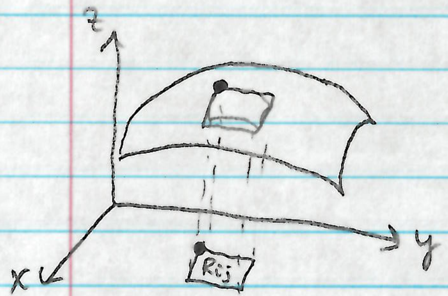
$$\bar{x} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA = \frac{1}{\left(\frac{k}{3}\right)}\int_0^{\pi/2}\int_0^1 k\,r^3\cos\theta\sin\theta\,dr\,d\theta$$
$$= 3\int_0^{\pi/2} \frac{r^4}{4}\cos\theta\sin\theta \Big|_0^1 d\theta = \frac{3}{4}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$
$$u = \sin\theta, \quad du = \cos\theta\,d\theta$$
$$= \frac{3}{4}\int_0^1 u\,du = \frac{3}{4}\cdot\frac{1}{2}u^2\Big|_0^1 = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA = \frac{1}{\left(\frac{k}{3}\right)}\int_0^{\pi/2}\int_0^1 k\,r^3\sin^2\theta\,dr\,d\theta$$
$$= 3\int_0^{\pi/2}\frac{r^4}{4}\sin^2\theta\Big|_0^1 d\theta = \frac{3}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta)\,d\theta$$
$$= \frac{3}{8}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} = \frac{3}{8}\left(\frac{\pi}{2} - 0 - 0 + 0\right) = \frac{3\pi}{16}$$

$$\boxed{\text{So } (\bar{x},\bar{y}) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)}$$

15.5 Surface Area

In calc 2, you saw how to find the surface area of a surface of revolution using a single integral. Now, we use double integrals to find the surface area of a surface $z = f(x,y)$ over a region D.



Given a surface $z=f(x,y)$, look just at the piece above a rectangle R_{ij}. Choosing a point in R_{ij}, the tangent plane to z at that point over R_{ij} will have a similar area to that of z over R_{ij}.

So $A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta T_{ij}$, where ΔT_{ij} is the area of the tangent plane to z at a point over R_{ij}.

To compute ΔT_{ij}, notice that the rectangle of the tangent plane is determined by the vectors

$\vec{a} = \langle \Delta x, 0, f_x(x_{ij}, y_{ij})\Delta x\rangle$, $\vec{b} = \langle 0, \Delta y, f_y(x_{ij}, y_{ij})\Delta y\rangle$,

where R_{ij} is determined by $\langle \Delta x, 0, 0\rangle$ and $\langle 0, \Delta y, 0\rangle$.

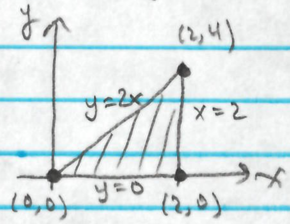
The area of the rectangle is $|\vec{a}\times\vec{b}|$

$$= |\langle -f_x(x_{ij}, y_{ij})\Delta x\Delta y, -f_y(x_{ij}, y_{ij})\Delta x \Delta y, \Delta x\Delta y\rangle|$$

$$= \sqrt{(f_x(x_{ij}, y_{ij}))^2 + (f_y(x_{ij}, y_{ij}))^2 + 1} \cdot \underbrace{\Delta x \Delta y}_{\Delta A}$$

So
$$\boxed{\begin{aligned} S(A) &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} \sqrt{(f_x(x_{ij}, y_{ij}))^2 + (f_y(x_{ij}, y_{ij}))^2 + 1}\,\Delta A \\ &= \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA \end{aligned}}$$

Ex 4. Find the surface area of the surface of the part of $2y+4z-x^2=5$ that lies above the triangle with vertices $(0,0)$, $(2,0)$, and $(2,4)$.



$4z = 5+x^2-2y$ so $z = \frac{5}{4}+\frac{1}{4}x^2-\frac{1}{2}y$

$\frac{\partial z}{\partial x} = \frac{1}{2}x$, $\frac{\partial z}{\partial y} = -\frac{1}{2}$

$$S(A) = \int_0^2\int_0^{2x} \sqrt{(\tfrac{1}{2}x)^2+(-\tfrac{1}{2})^2+1}\,dy\,dx$$

$$= \int_0^2\int_0^{2x} \sqrt{\tfrac{1}{4}x^2+\tfrac{5}{4}}\,dy\,dx$$

$$= \int_0^2 \sqrt{\tfrac{1}{4}x^2+\tfrac{5}{4}}\,y\Big|_0^{2x}\,dx$$

$$= \int_0^2 2x\sqrt{\tfrac{1}{4}x^2+\tfrac{5}{4}}\,dx$$

$u = \frac{1}{4}x^2+\frac{5}{4}$, $du = \frac{1}{2}x\,dx$

$$= 4\int_{5/4}^{9/4} u^{1/2}\,du$$

$$= 4\cdot\tfrac{2}{3}u^{3/2}\Big|_{5/4}^{9/4} = \tfrac{8}{3}\left(\left(\tfrac{9}{4}\right)^{3/2}-\left(\tfrac{5}{4}\right)^{3/2}\right)$$

$$= \tfrac{8}{3}\left(\tfrac{27}{8}-\tfrac{5\sqrt{5}}{8}\right)$$

$$= \tfrac{1}{3}(27-5\sqrt{5})$$

$$= \boxed{9-\tfrac{5}{3}\sqrt{5}}$$