MA 261 - Lesson 19

The Fundamental Theorem for Line Integrals (16.3)

Recall: A vector field $\vec{F}$ is called conservative if $\vec{F} = \nabla f$ for some function f, which is called a potential function for $\vec{F}$.

Conservative vector fields are important since ∇f is like a "derivative" of f, and remember the Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

We get the same thing for line integrals of conservative vector fields.

Fundamental Theorem for Line Integrals (FTfLI)
Let C be a smooth curve given by the vector function $\vec{r}(t)$, $a \le t \le b$. Let f be a differentiable function whose gradient ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof.

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) dt \\ &= \int_a^b \frac{d}{dt}\big(f(\vec{r}(t)) \big)\,dt \quad \text{(Chain Rule)} \\ &= f(\vec{r}(b)) - f(\vec{r}(a)) \\ &\qquad \text{(Fund. Thm. Calc)}\end{aligned}$$

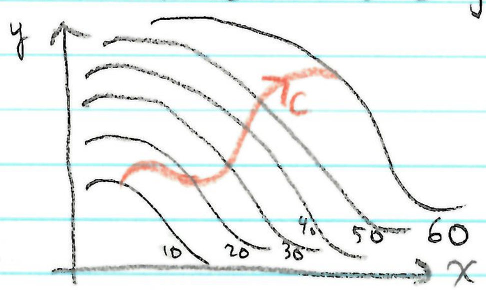
(Same argument applies for 3 variables)

Notice: If $\vec{F}$ is a conservative vector field, we get that $\int_C \vec{F} \cdot d\vec{r}$ is path independent!
i.e., if C_1 and C_2 are smooth curves both starting at A and ending at B, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = f(B) - f(A) = \int_{C_2} \vec{F} \cdot d\vec{r},$$
where $\vec{F} = \nabla f$.

This is way false is $\vec{F}$ is not conservative (see Lesson 18 Examples 1 and 2)

Ex 1. Compute $\int_C \nabla f \cdot d\vec{r}$, where $C = \vec{r}(t)$ is graphed below, ∇f is continuous, and several level curves are graphed below.



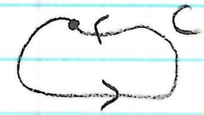
Notice: C is a smooth curve and ∇f is continuous, hence, by the FTfLI,

$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$, where B is the endpoint of C and A is the start point of C.

Notice: C ends on the level curve where $f = 60$ and starts on the level curve where $f = 10$ so

$$\int_C \nabla f \cdot d\vec{r} = 60 - 10 = \boxed{50}$$

Another implication of FTfLI is that if C is a closed path (starting and ending at the same point), then



$$\int_C \vec{F}\cdot d\vec{r} = 0, \text{ whenever } \vec{F} \text{ is conservative}$$

This is again way false if $\vec{F}$ is not conservative (see Lesson 18, Ex. 5)

Conservative Vector Fields

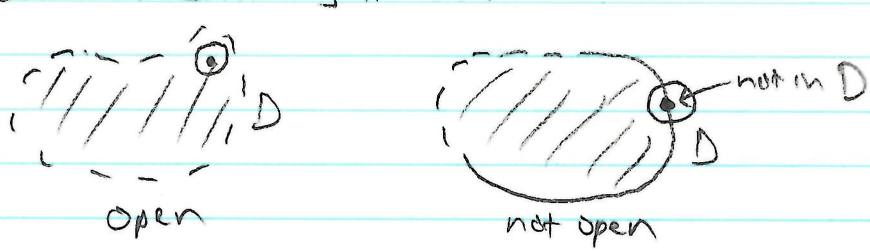
So the natural question arises: How can we tell whether $\vec{F}$ is conservative or not?

One answer is the above:

Theorem. Suppose $\vec{F}$ is continuous on an open connected region D. If $\int_C \vec{F}\cdot d\vec{r}$ is path independent in D, then $\vec{F}$ is conservative on D. Equivalently, if $\int_C \vec{F}\cdot d\vec{r} = 0$ for all closed curves in D, then $\vec{F}$ is conservative on D.

(Proof in textbook)

A region D in $\mathbb{R}^2$ or $\mathbb{R}^3$ is open if all points "nearby" are also in D (i.e., if every point x has a disk around it entirely in D).

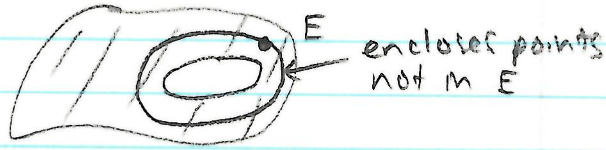


A region D is <u>connected</u> if for any pair of points P and Q in D, there is a path in D connecting P and Q.

connected connected not connected

A region D is called <u>simply connected</u> if it is connected and if any closed curve in D encloses only points in D.

Above, D is simply connected, but E is not



In the next theorem, open simply connected regions are important. Note: $\mathbb{R}^2$, $\mathbb{R}^3$ are open simply connected regions.

Clairaut's Theorem can tell us if a vector field is not conservative.

If $\vec{F} = P\hat{i} + Q\hat{j}$, then if $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, $\vec{F}$ cannot be conservative!

(If $\vec{F} = \nabla f$, then $P = f_x$, $Q = f_y$,
so $\frac{\partial P}{\partial y} = f_{xy} = f_{yx} = \frac{\partial Q}{\partial x}$)

We get a converse on open simply connected regions!

Theorem. Let $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ be a vector field on an open simply connected region D. Suppose P and Q have continuous first-order partial derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D. Then $\vec{F}$ is conservative.

Ex 2. Determine whether $\vec{F}$ is conservative

(a) $\vec{F}(x,y) = 3xy\,\hat{\imath} + x^2y\,\hat{\jmath}$

$$\frac{\partial P}{\partial y} = 3x, \quad \frac{\partial Q}{\partial x} = 2xy$$

$3x \neq 2xy$, so by Clairaut's Theorem, $\vec{F}$ is $\boxed{\text{not}}$ conservative

(b) $\vec{F}(x,y) = (2x+2y)\hat{\imath} + (2x+3y^2-\sin y)\hat{\jmath}$

$$\frac{\partial P}{\partial y} = 2, \quad \frac{\partial Q}{\partial x} = 2$$

The domain of $\vec{F}$ in $\mathbb{R}^2$ which is an open simply connected region, so $\vec{F}$ $\boxed{\text{is conservative.}}$

So how do we find a potential function f?

Notice, if $\vec{F} = P\hat{\imath} + Q\hat{\jmath} = \nabla f$, then $f_x = P$ and $f_y = Q$.

So $f = \int P\,dx$ (remember to include an arbitrary function $g(y)$)

Then take $\frac{\partial}{\partial y}(f)$ and set it equal to Q to determine $g(y)$.

Ex 3. Find all potential functions for
$\vec{F}(x,y) = (2x+2y)\hat{\imath} + (2x+3y^2-\sin y)\hat{\jmath}$

Saw $\vec{F}$ conservative in Ex 2, so $P = 2x+2y = f_x$

So $f = \int f_x\,dx = \int (2x+2y)\,dx = x^2+2xy+g(y)$

Hence, $f_y = 2x + g'(y)$

But $Q = 2x+3y^2-\sin y = f_y$ too

$2x+3y^2-\sin y = 2x+g'(y)$

$3y^2 - \sin y = g'(y)$

$g(y) = \int g'(y)\,dy = \int (3y^2-\sin y)\,dy = y^3+\cos y + K$

so $\boxed{f = x^2+2xy+y^3+\cos y + K}$

Ex 4. $\vec{F}(x,y,z) = (3x^2+yz)\hat{\imath} + (xz+e^y)\hat{\jmath} + (xy+2z)\hat{k}$ is conservative. Find a potential function.

Since $\vec{F}$ is conservative, $P = f_x$, $Q = f_y$, $R = f_z$

Hence, $f = \int (3x^2+yz)\,dx = x^3+xyz+g(y,z)$

$f_y = xz + \frac{\partial g}{\partial y}$ and also $= Q = xz+e^y$

so $g_y = e^y \Rightarrow g(y,z) = \int e^y\,dy = e^y + h(z)$

So $f = x^3+xyz+e^y+h(z)$

$f_z = xy + h'(z)$ and also $= R = xy+2z$

so $h'(z) = 2z \Rightarrow h(z) = \int 2z\,dz = z^2+K$

So $f = x^3+xyz+e^y+z^2+K$

Can choose $K=0$, so $\boxed{f = x^3+xyz+e^y+z^2}$

Ex 5. Find the work done by the vector field $\vec{F}(x,y) = (ye^x + 2x)\hat{\imath} + (e^x - \sin y)\hat{\jmath}$ on an object moving from $(0,0)$ to $(1,\pi)$.

$W = \int_C \vec{F} \cdot d\vec{r}$

Is $\vec{F}$ conservative? $\frac{\partial P}{\partial y} = e^x$, $\frac{\partial Q}{\partial x} = e^x$

so yes!

$f = \int P\,dx = \int (ye^x + 2x)\,dx = ye^x + x^2 + g(y)$

$f_y = e^x + g'(y)$ and also $= Q = e^x - \sin y$

so $g'(y) = -\sin y \Rightarrow g(y) = \cos y$

$f = ye^x + x^2 + \cos y$

By FTfLI, $\int_C \vec{F} \cdot d\vec{r} = f(1,\pi) - f(0,0)$

$= [\pi e^1 + (1)^2 + \cos(\pi)] - [0 + 0 + \cos(0)]$

$= \pi e + 1 - 1 - 1$

$= \boxed{\pi e - 1}$ units of work

Ex 6. Compute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = (1 + ye^{xy})\hat{\imath} + (3 + xe^{xy})\hat{\jmath}$ where C is the line segment from $(1,1)$ to $(0,0)$.

Is $\vec{F}$ conservative? $\frac{\partial P}{\partial y} = yxe^{xy} + e^{xy}$, $\frac{\partial Q}{\partial x} = xye^{xy} + e^{xy}$ yes!

$f = \int P\,dx = \int (1 + ye^{xy})\,dx = x + e^{xy} + g(y)$

$f_y = xe^{xy} + g'(y)$ also $= 3 + xe^{xy}$

$\Rightarrow g'(y) = 3 \Rightarrow g(y) = 3y$

$f = x + e^{xy} + 3y$

By FTfLI, $\int_C \vec{F} \cdot d\vec{r} = f(0,0) - f(1,1)$

$= (0 + 1 + 0) - (1 + e + 3)$

$= 1 - (e + 4)$

$= \boxed{-e - 3}$

Why "conservative"?

Suppose F is a conservative force field moving an object from point A to point B along a smooth curve.

The work done is $W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

(By Newton's Second Law, $\vec{F}(\vec{r}(t)) = m\vec{a}(t) = m\vec{r}\,''(t)$)

$$= \int_a^b m\vec{r}\,''(t) \cdot \vec{r}\,'(t)\,dt$$

$$= m\int_a^b \tfrac{1}{2}\tfrac{d}{dt}\left[\vec{r}\,'(t) \cdot \vec{r}\,'(t)\right]dt \quad \text{(product rule)}$$

$$= \tfrac{m}{2}\int_a^b \tfrac{d}{dt}\left[|\vec{r}\,'(t)|^2\right]dt$$

$$= \tfrac{m}{2}|\vec{r}\,'(t)|^2\Big|_a^b = \tfrac{m}{2}|\vec{r}\,'(b)|^2 - \tfrac{m}{2}|\vec{r}\,'(a)|^2$$

$$= \tfrac{1}{2}m|\vec{v}(b)|^2 - \tfrac{1}{2}m|\vec{v}(a)|^2$$

$$= K(B) - K(A)$$

where K represents kinetic energy.

The Potential energy P is given by $-f$, so since $\vec{F}$ is conservative, $\vec{F} = \nabla f = -\nabla P$.

$W = \int_C \vec{F} \cdot d\vec{r} = \int_C -\nabla P \cdot d\vec{r} = -P(B) + P(A)$
by FTfLI

So $-P(B) + P(A) = W = K(B) - K(A)$

$$P(A) + K(A) = P(B) + K(B)$$

Hence, if $\vec{F}$ is a conservative vector field, then energy is conserved.

(Law of Conservation of Energy holds)