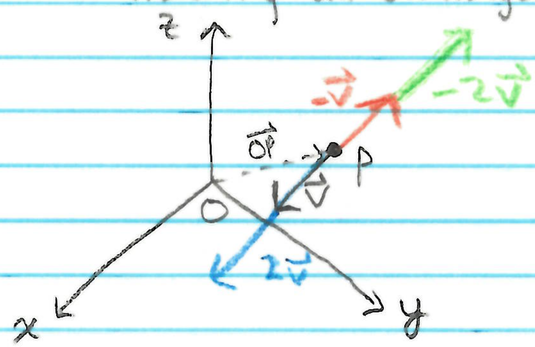
MA 261 – Lesson 2

Lines, Planes, Cylinders, and Quadric Surfaces

12.5 – Lines and Planes

We know from geometry that a line is determined by two points. But two points also determine a vector or direction. So a line can be determined by a point and a vector, by placing the vector's tail at the point and considering all scalings of that vector.



Vector Equations of a Line:
Given a point $P(a,b,c)$ and a vector $\vec{v} = \langle \alpha, \beta, \gamma \rangle$, the line passing through the point P in the direction of $\vec{v}$ can be given by

$$\vec{r} = \overrightarrow{OP} + t\vec{v} = \langle a,b,c \rangle + t\langle \alpha, \beta, \gamma \rangle = \langle a+\alpha t, b+\beta t, c+\gamma t \rangle$$

This is called a vector equation of the line. Vector equations are not unique, since we could be given a different point P or vector $\vec{v}$.

Notice that the first, second, and third coordinates of the vector equation correspond to the x-, y-, and z-coordinates of the points. Treating t as a parameter, we get parametric equations.

Parametric Equations of a Line:

Given a vector equation $\vec{r} = \langle a+\alpha t, b+\beta t, c+\gamma t \rangle$, we get parametric equations

$$x = a+\alpha t, \quad y = b+\beta t, \quad z = c+\gamma t$$

of the line, for all values of t in $\mathbb{R}$.

Next, notice we can solve each of these for t to eliminate the parameter. We get

$$t = \frac{x-a}{\alpha}, \quad t = \frac{y-b}{\beta}, \quad t = \frac{z-c}{\gamma}$$

Symmetric Equations of a Line:

Given a set of parametric equations of a line, as above, and eliminating the parameter, we get

$$\frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma}$$

which are called symmetric equations of the line.

If any of α, β, or γ are 0, we can have multiple equations, such as $x = a$, $\frac{y-b}{\beta} = \frac{z-c}{\gamma}$

The numbers α, β, and γ in the symmetric equations are called the <u>direction numbers</u> of the line since they determine a direction vector for the line.

<u>Ex 1</u>. Find a vector equation, parametric equation, symmetric equations, and the direction numbers for the line passing through the points $P(1,2,-1)$ and $Q(3,0,1)$.

Get vector $\vec{PQ} = \langle 3-1, 0-2, 1-(-1) \rangle = \langle 2,-2,2 \rangle =: \vec{v}$

vector equation:

$$\vec{r} = \vec{OP} + t\vec{v} = \langle 1,2,-1 \rangle + t\langle 2,-2,2 \rangle = \langle 1,2,-1 \rangle + \langle 2t,-2t,2t \rangle$$

$$\boxed{\vec{r} = \langle 1+2t, 2-2t, 2+2t \rangle}$$

Parametric equations:

$$\boxed{x = 1+2t,\ y = 2-2t,\ z = 2+2t}$$

Symmetric equations

$$t = \boxed{\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z-2}{2}}$$

direction numbers: $\boxed{2,-2,2}$

If we want only a line <u>segment</u>, we can restrict the values of t.

A nice way to get a vector function for the line segment starting at a point P and ending at a point Q is to take

$$\vec{r} = (1-t)\vec{OP} + t\,\vec{OQ} \quad \text{for } 0 \le t \le 1$$

Notice; when $t=0$, we get $\vec{OP}$, which is at the point P

when $t=1$, we get $\vec{OQ}$, which is at the point Q.

and we also have a continuous segment of the line containing P and Q, so it must be the line segment starting at P and ending at Q.

Ex 2. Find a vector equation for the line segment starting at $(2,1,-1)$ and ending at $(4,3,-2)$.

$$\vec{r} = (1-t)\langle 2,1,-1\rangle + t\langle 4,3,-2\rangle, \quad 0 \le t \le 1$$
$$= \langle 2-2t, 1-t, -1+t\rangle + \langle 4t, 3t, -2t\rangle, \quad 0 \le t \le 1$$
$$= \boxed{\langle 2+2t, 1+2t, -1-t\rangle, \quad 0 \le t \le 1}$$

Two lines ℓ_1 and ℓ_2 are parallel if their direction vectors are parallel. ℓ_1 and ℓ_2 are called skew lines if they do not intersect and are not parallel.

We know from geometry that two non-skew lines determine a plane. In particular, two intersecting lines determine a plane (for parallel lines, take any line intersecting both of them to get intersecting lines). Notice that two intersecting lines determine a point (of intersection) and a direction orthogonal to both lines. Notice that this orthogonal direction is orthogonal to all vectors living in that plane.

We call a vector orthogonal to a plane a normal vector to the plane and denote it $\vec{n}$.

Let $Q(x,y,z)$ be an arbitrary point in the plane and $P(a,b,c)$ be the specific known point and $\vec{n} = \langle \alpha, \beta, \gamma\rangle$ be a normal vector.
Then $\vec{PQ} = \langle x-a, y-b, z-c\rangle$ is in the plane, so
$0 = \vec{n}\cdot\vec{PQ} = \alpha(x-a) + \beta(y-b) + \gamma(z-c)$

Hence, the equation of a plane containing $P(a,b,c)$ with a normal vector $\vec{n} = \langle \alpha, \beta, \gamma\rangle$ is
$$\alpha(x-a) + \beta(y-b) + \gamma(z-c) = 0$$

Ex 3. At what point does the line with vector equation $\vec{r} = \langle 3t, 1-2t, -2+t \rangle$ intersect the plane containing point $P(1,0,-2)$ and having normal vector $\vec{n} = \langle 1,-1,1 \rangle$?

Find equation of plane: $1(x-1) - 1(y-0) + 1(z+2) = 0$

$$x - 1 - y + z + 2 = 0$$

$$x - y + z = -1$$

The line has parametric equations

$$x = 3t, \quad y = 1-2t, \quad z = -2+t$$

So the point of intersection occurs at the t-value satisfying

$$3t - (1-2t) + (-2+t) = -1$$

$$3t - 1 + 2t - 2 + t = -1$$

$$6t = 2$$

$$t = \frac{1}{3}$$

So the point is given by $\left(3\left(\frac{1}{3}\right), 1-2\left(\frac{1}{3}\right), -2+\left(\frac{1}{3}\right)\right)$

$$= \boxed{\left(1, \frac{1}{3}, -\frac{5}{3}\right)}$$

Consider two planes P_1 and P_2 with normal vectors $\vec{n}_1$ and $\vec{n}_2$, respectively.

- P_1 and P_2 are parallel if $\vec{n}_1$ and $\vec{n}_2$ are parallel
- P_1 and P_2 are orthogonal if $\vec{n}_1$ and $\vec{n}_2$ are orthogonal
- The angle between P_1 and P_2 is the angle between $\vec{n}_1$ and $\vec{n}_2$.

Ex 4. Find a vector equation for the line of intersection of the planes $x - y + 3z = -5$ and $2x - 4y + z = -9$.

We need a point on the line and a direction vector.

To get a point, can fix $x = 0$ and solve for y and z.

$$\begin{cases} -y + 3z = -5 \\ -4y + z = -9 \end{cases} \Rightarrow y = 2 \text{ and } z = -1$$

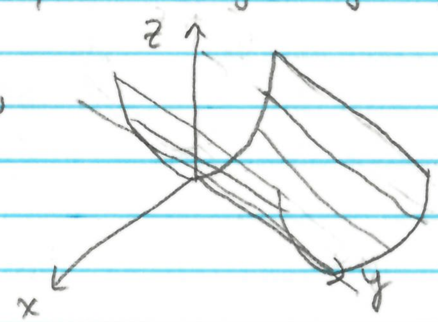
So $(0, 2, -1)$ is a point on the line.

(Ex 4 continued): The planes have normal vectors
$\vec{n_1} = \langle 1, -1, 3 \rangle$ and $\vec{n_2} = \langle 2, -4, 1 \rangle$.
The line must line on both planes, so must lie in a direction orthogonal to both $\vec{n_1}$ and $\vec{n_2}$.
Take $\vec{v} = \vec{n_1} \times \vec{n_2} = \langle 11, 5, -2 \rangle$
Thus, the line of intersection has vector equation
$\vec{r} = \langle 0, 2, -1 \rangle + t\langle 11, 5, -2 \rangle = \boxed{\langle 11t, 2+5t, -1-2t \rangle}$

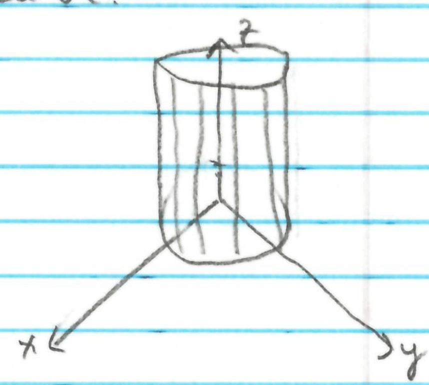
12.6 Cylinders and Quadric Surfaces

A cylinder is a surface consisting of all lines (called rulings) that are parallel to a given line and pass through a give plane curve.

e.g.,



a parabolic cylinder with rulings parallel to the y-axis passing through the curve
$z = x^2,\ y = 0$
equation: $z = x^2$

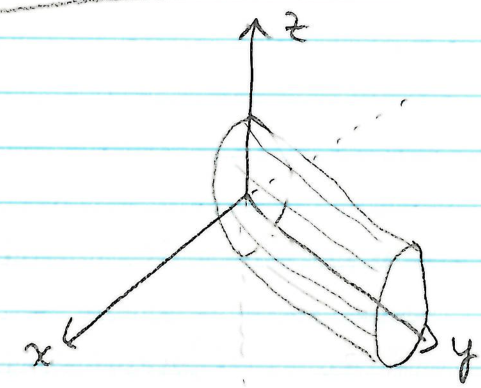


a circular cylinder with rulings parallel to the z-axis passing through the curve
$x^2 + y^2 = 1,\ z = 0$
equation: $x^2 + y^2 = 1$

If you are given an equation in which a single variable is missing, the surface it represents is a cylinder with rulings parallel to that variable's axis.

Ex 5. Describe and sketch the surface with equation
$x^2 + \frac{z^2}{4} = 1$

"y" is missing so we have a cylinder.
$x^2 + \frac{z^2}{4} = 1$, $y = 0$ is an ellipse on the xz-plane.
So we have an elliptic cylinder with rulings parallel to the y-axis passing through the curve
$x^2 + \frac{z^2}{4} = 1, y = 0$



Notice that the rulings of a cylinder are intersections of the surface with a plane. This technique can help us identify and sketch other surfaces.

A trace is the intersection of a surface with a plane of the form $x = k$, $y = k$, or $z = k$.

Traces help us in identifying quadric surfaces, which are surfaces whose equations have terms of degree 2 or lower.

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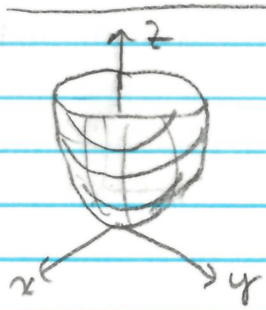
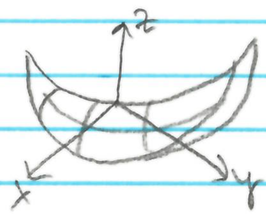
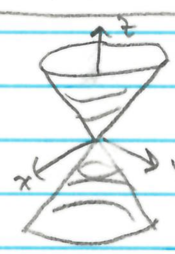
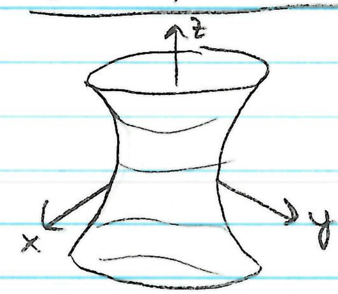
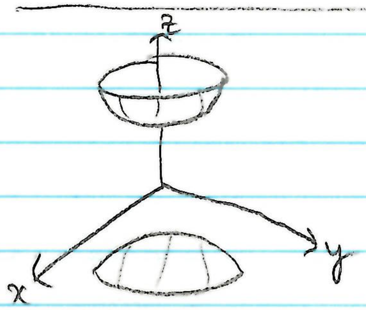
Classification of Quadric Surfaces:

The following table gives all quadric surfaces with equations in standard form, symmetric with respect to the z-axis. To get it symmetric to a different axis, you mult adjust the equation.

Sketch/Name	Traces	Equation
 Ellipsoid	All traces are ellipses	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ If $a=b=c=1$, the ellipsoid is a sphere
 Elliptic Paraboloid	Horizontal traces are ellipses (1 direction) Vertical traces are parabolas (2 directions)	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ A variable raised to the first power indicates the axis of the paraboloid
 Hyperbolic Paraboloid	Horizontal traces are hyperbolas (1 direction) Vertical traces are parabolas (2 directions)	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$
 Cone	Horizontal traces are ellipses (1 direction) Vertical traces are hyperbolas if $k \neq 0$ and pairs of lines if $k=0$ (2 directions)	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

Sketch/Name	Traces	Equation
 Hyperboloid of one sheet	Horizontal traces are ellipses (1 dir.) Vertical traces are hyperbolas (2 dir.)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ The axis of symmetry corresponds to variable w/ negative coefficient.
 Hyperboloid of two sheets	Horizontal traces are ellipses if $k > c$ or $k < -c$ (1 dir.) Vertical traces are hyperbolas (2 dir.)	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ The two minus signs indicate two sheets.

Ex 6. Use traces to identify the surface

$$z^2 - 4x^2 + 8x - 4y^2 - 16y = 16$$

x=k traces: $z^2 - 4y^2 - 16y = 16 + 4k^2 - 8k$

$$z^2 - 4(y^2 + 4y + \underline{4}) = 16 + 4k^2 - 8k - 4\cdot\underline{4}$$

$$z^2 - 4(y+2)^2 = 4k^2 - 8k$$

$$\frac{z^2}{4k^2 - 8k} - \frac{(y+2)^2}{k^2 - 8k} = 1 \quad \text{hyperbolas}$$

y=k traces: $z^2 - 4x^2 + 8x = 16 + 4k^2 + 16k$

$$z^2 - 4(x-1)^2 = 12 + 4k^2 + 16k$$

$$\frac{z^2}{12 + 4k^2 + 16k} - \frac{(x-1)^2}{12 + 4k^2 + 16k} = 1 \quad \text{hyperbolas}$$

z=k traces: $-4(x-1)^2 - 4(y+2)^2 = 1 - k^2$

$$(x-1)^2 + (y+2)^2 = \frac{1-k^2}{-4}$$ circles (ellipses) provided $1 - k^2 < 0 \iff k > 1$ or $k < -1$

by checking traces, must be a $\boxed{\text{hyperboloid of two sheets}}$

Ex 7. Classify the surface by putting the equation in standard form: $36x^2 - 9y^2 + 4z^2 - 8z + 4 = 0$

$$36x^2 - 9y^2 + 4(z^2 - 2z + \underline{1}) = -4 + 4 \cdot \underline{1}$$
$$36x^2 - 9y^2 + 4(z-1)^2 = 0$$
$$36x^2 + 4(z-1)^2 = 9y^2$$

Divide by 36

$$\frac{x^2}{1} + \frac{(z-1)^2}{9} = \frac{y^2}{4}$$

standard form matches that of a

$\boxed{\text{cone symmetric to the } y\text{-axis}}$