MA 261 - Lesson 23

Areas of Parametric Surfaces and Surface Integrals

16.6 Areas of Parametric Surfaces

When we found the surface area of a surface $z = f(x,y)$ in lesson 13, we used the tangent plane to develop a Riemann sum to get the surface area. The same idea will apply to parametric surfaces.

Tangent Planes

Given a parametric surface $\vec{r}(u,v)$ and a point (a,b), notice that we can use the grid lines of the surface to find tangent vectors.

In particular, $\vec{r}_u = \frac{\partial}{\partial u}\vec{r}(u,v)$ and $\vec{r}_v = \frac{\partial}{\partial v}\vec{r}(u,v)$ are tangent vectors along the grid lines of the surface. Hence, $\vec{r}_u \times \vec{r}_v$ is a normal vector for the tangent plane.

Ex 1. Find an equation for the tangent plane to the parametric surface $x = u^2+1$, $y = v^3+1$, $z = u+v$ at the point $(5,2,3)$.

$\vec{r}_u = \langle 2u, 0, 1\rangle$, $\vec{r}_v = \langle 0, 3v^2, 1\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 0 & 1 \\ 0 & 3v^2 & 1 \end{vmatrix} = (-3v^2)\hat{i} - (2u)\hat{j} + (6uv^2)\hat{k}$$

The point $(5,2,3)$ occurs when

$5 = x = u^2+1$,	$2 = y = v^3+1$,	$3 = z = u+v$
$u^2 = 4$	$v^3 = 1$	$3 = u+1$
$u = \pm 2$	$v = 1$	$u = 2$, not -2

Thus, at $(5,2,3)$ the normal vector is
$\langle -3(1)^2, -2(2), 6(2)(1)^2 \rangle = \langle -3, -4, 12 \rangle$

So the plane has equation
$-3(x-5) - 4(y-2) + 12(z-3) = 0$
or $-3x + 15 - 4y + 8 + 12z - 36 = 0$
$\boxed{-3x - 4y + 12z = 13}$

Surface Areas

Given a parametric surface $\vec{r}(u,v)$, $a \le u \le b$, $c \le v \le d$, we can split the surface into small rectangles, like in lesson 13. For the same reason, we get that the surface area is

$$\boxed{S(A) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA, \text{ where } D \text{ is the rectangle in the } uv\text{-plane}, \ a \le u \le b, \ c \le v \le d.}$$

Ex 2. Find the area of the surface $x = z^2 + y$ that lies between the planes $y=0$, $y=2$, $z=0$, $z=2$.

Get the parameterization $\vec{r}(y,z) = \langle z^2+y, y, z \rangle$, $0 \le y \le 2$, $0 \le z \le 2$.

$\vec{r}_y = \langle 1, 1, 0 \rangle$, $\vec{r}_z = \langle 2z, 0, 1 \rangle$

$$\vec{r}_y \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 2z & 0 & 1 \end{vmatrix} = 1\hat{i} - 1\hat{j} + (-2z)\hat{k}$$

$$|\vec{r}_y \times \vec{r}_z| = \sqrt{1 + 1 + 4z^2} = \sqrt{4z^2+2} = \sqrt{2}\sqrt{2z^2+1}$$

$$S(A) = \int_0^2 \int_0^2 \sqrt{2} \cdot \sqrt{2z^2+1} \, dy \, dz$$

$$= 2\sqrt{2} \int_0^2 \sqrt{2z^2+1} \, dz$$

Trig sub: Want to use $\tan^2\theta + 1 = \sec^2\theta$

Let $\sqrt{2}\,z = \tan\theta$ (then $2z^2+1 = \tan^2\theta + 1 = \sec^2\theta$)

$\sqrt{2}\,dz = \sec^2\theta\, d\theta$

when $z=0$, $\sqrt{2}\cdot 0 = \tan\theta \Rightarrow 0 = \tan\theta \Rightarrow \theta = 0$

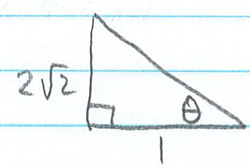
when $z=2$, $\sqrt{2}\cdot 2 = \tan\theta \Rightarrow \theta = \tan^{-1}(2\sqrt{2})$

$$2\int_0^{\tan^{-1}(2\sqrt{2})} \sqrt{\sec^2\theta}\,\sec^2\theta\, d\theta$$

$$= 2\int_0^{\tan^{-1}(2\sqrt{2})} \sec^3\theta\, d\theta$$

$$= 2\left[\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\tan^{-1}(2\sqrt{2})}$$

$$= \sec(\tan^{-1}(2\sqrt{2}))\cdot 2\sqrt{2} + \ln|\sec(\tan^{-1}(2\sqrt{2})) + 2\sqrt{2}|$$
$$- 0 - \ln(1)$$



Letting $\theta = \tan^{-1}(2\sqrt{2})$

$\tan\theta = 2\sqrt{2} = \dfrac{\text{opposite}}{\text{adjacent}}$

By Pythagorean Theorem, $(2\sqrt{2})^2 + 1^2 = \text{hyp}^2$

$9 = \text{hyp}^2 \Rightarrow \text{hyp} = 3$

$\sec(\tan^{-1}(2\sqrt{2})) = \sec(\theta) = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{3}{1} = 3$

so... $3\cdot 2\sqrt{2} + \ln|3 + 2\sqrt{2}|$

$$= \boxed{6\sqrt{2} + \ln(3 + 2\sqrt{2})}$$

We can recover the formula for the surface area of a function $z = f(x,y)$ by parameterizing as $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$ over D.

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = f_x \hat{\imath} - f_y \hat{\jmath} + \hat{k}$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{(f_x)^2 + (f_y)^2 + 1}$$

So $S(A) = \iint_D |\vec{r}_x \times \vec{r}_y| \, dA = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2} \, dA$

Ex 3. Find the area of the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \le x \le 1$, $0 \le y \le 1$

$\frac{\partial z}{\partial x} = \sqrt{x}$, $\frac{\partial z}{\partial y} = \sqrt{y}$ so

$$\int_0^1 \int_0^1 \sqrt{1 + x + y} \, dx \, dy$$

$$= \int_0^1 \frac{2}{3}(1 + x + y)^{3/2} \Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{2}{3}(2+y)^{3/2} - \frac{2}{3}(1+y)^{3/2} \right) dy$$

$$= \frac{2}{3} \cdot \frac{2}{5}(2+y)^{5/2} - \frac{2}{3} \cdot \frac{2}{5}(1+y)^{5/2} \Big|_0^1$$

$$= \frac{4}{15}\left(3^{5/2} - 2^{5/2} - 2^{5/2} + 1^{5/2} \right)$$

$$= \boxed{\frac{4}{15}\left(3^{5/2} - 2^{7/2} + 1 \right)}$$

16.7 Surface Integrals of a Function

Line integrals were where we took an integral of a 2-variable function over a curve instead of over a region. Similarly, for a surface integral, we take an integral of a 3-variable function over a surface instead of over a 3-d region.

Recall that for $\int_C f(x,y)\,ds$, we needed to include the ds, which was the differential of arc length. For the same reason, for the surface integral $\iint_S f(x,y,z)\,dS$, we include dS, the differential of surface area of the surface, which is $|\vec{r}_u \times \vec{r}_v|\,dA$, where the surface has parametric representation $\vec{r}(u,v)$ over the region D in the uv-plane.

Just like with line integrals

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}\,dt$$

we get

$$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v),y(u,v),z(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

Ex 4. Compute $\iint_S y\, dS$, where S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \le x \le 1$, $0 \le y \le 1$.

From ex 3, saw $\vec{r}(x,y) = \langle x, y, \frac{2}{3}(x^{3/2}+y^{3/2}) \rangle$, $0 \le x \le 1$, $0 \le y \le 1$ with $|\vec{r}_x \times \vec{r}_y| = \sqrt{1+x+y}$

so $$\int_0^1 \int_0^1 y\sqrt{1+x+y}\, dx\, dy$$

$$= \int_0^1 \frac{2}{3} y (1+x+y)^{3/2} \Big|_{x=0}^{x=1} dy$$

$$= \frac{2}{3} \int_0^1 \left(y(2+y)^{3/2} - y(1+y)^{3/2} \right) dy$$

$u = 2+y$, so $y = u-2$, $du = dy$ $\qquad$ $v = 1+y$, so $y = v-1$, $dv = dy$

$$= \frac{2}{3}\left[\int_2^3 (u-2)u^{3/2}\, du - \int_1^2 (v-1)v^{3/2}\, dv \right]$$

$$= \frac{2}{3}\left[\int_2^3 (u^{5/2} - 2u^{3/2})\, du - \int_1^2 (v^{5/2} - v^{3/2})\, dv \right]$$

$$= \frac{2}{3}\left[\left(\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2}\right)\Big|_2^3 - \left(\frac{2}{7}v^{7/2} - \frac{2}{5}v^{5/2}\right)\Big|_1^2 \right]$$

$$= \frac{2}{3}\left[\frac{2}{7}(3)^{7/2} - \frac{4}{5}(3)^{5/2} - \frac{2}{7}(2)^{7/2} + \frac{4}{5}(2)^{5/2} - \frac{2}{7}(2)^{7/2} + \frac{2}{5}(2)^{5/2} + \frac{2}{7} - \frac{2}{5} \right]$$

$$= \frac{2}{3}\left[\frac{54}{7}\sqrt{3} - \frac{36}{5}\sqrt{3} - \frac{16}{7}\sqrt{2} + \frac{16}{5}\sqrt{2} - \frac{16}{7}\sqrt{2} + \frac{8}{5}\sqrt{2} - \frac{4}{35} \right]$$

$$= \frac{2}{3}\left[\frac{18}{35}\sqrt{3} + \frac{8}{35}\sqrt{2} - \frac{4}{35} \right]$$

$$= \boxed{\frac{4}{105}\left[9\sqrt{3} + 4\sqrt{2} - 2 \right]}$$

Ex 5. Compute $\iint_S y^2 z^2 \, dS$ where S is the part of the cone $y = \sqrt{x^2+z^2}$ given by $0 \le y \le 5$

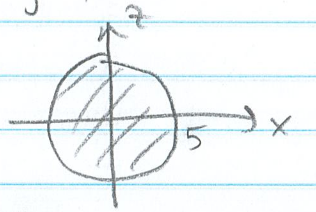
$\vec{r}(x,z) = \langle x, \sqrt{x^2+z^2}, z \rangle, \quad 0 \le \sqrt{x^2+z^2} \le 5$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & \frac{x}{\sqrt{x^2+z^2}} & 0 \\ 0 & \frac{z}{\sqrt{x^2+z^2}} & 1 \end{vmatrix} = \frac{x}{\sqrt{x^2+z^2}}\hat{\imath} - \hat{\jmath} + \frac{z}{\sqrt{x^2+z^2}}\hat{k}$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{x^2}{x^2+z^2} + 1 + \frac{z^2}{x^2+z^2}} = \sqrt{2}$$

$= \sqrt{2} \iint_D \left(\sqrt{x^2+z^2}\right)^2 z^2 \, dA$ where D is $0 \le \sqrt{x^2+z^2} \le 5$

Using polar coordinates on the xz-plane



$r^2 = x^2+z^2, \quad x = r\cos\theta, \quad z = r\sin\theta$

$$\sqrt{2}\int_0^{2\pi}\int_0^5 r^2 (r\sin\theta)^2 \cdot r \, dr \, d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^5 r^5 \sin^2\theta \, dr \, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1}{6} r^6 \left(\frac{1}{2}(1-\cos 2\theta)\right)\Big|_{r=0}^{r=5} d\theta$$

$$= \frac{15625\sqrt{2}}{12}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta$$

$$= \frac{15625\sqrt{2}}{12}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$\frac{15625\sqrt{2}}{12}(2\pi - 0 - 0 + 0)$$

$$\boxed{\frac{15625\sqrt{2}}{6}\pi}$$

Ex 6. Compute $\iint_S (x^2z + y^2z)\,dS$, where S is the hemisphere $x^2+y^2+z^2=9$, $z \le 0$.

Using spherical coordinates, the hemisphere is given by

$\rho = 3$, $0 \le \theta \le 2\pi$, $\frac{\pi}{2} \le \phi \le \pi$

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$\vec{r}(\theta,\phi) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi \rangle$,

$0 \le \theta \le 2\pi$, $\frac{\pi}{2} \le \phi \le \pi$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin\phi\sin\theta & 3\sin\phi\cos\theta & 0 \\ 3\cos\phi\cos\theta & 3\cos\phi\sin\theta & -3\sin\phi \end{vmatrix}$$

$$= (-9\sin^2\phi\cos\theta)\hat{i} - (9\sin^2\phi\sin\theta)\hat{j} + (-9\sin\phi\cos\phi\sin^2\theta - 9\sin\phi\cos\phi\cos^2\theta)\hat{k}$$

$$= (-9\sin^2\phi\cos\theta)\hat{i} - 9\sin^2\phi\sin\theta\,\hat{j} - 9\sin\phi\cos\phi\,\hat{k}$$

$$|\vec{r}_\theta \times \vec{r}_\phi| = \sqrt{81\sin^4\phi\cos^2\theta + 81\sin^4\phi\sin^2\theta + 81\sin^2\phi\cos^2\phi}$$

$$= \sqrt{81\sin^4\phi + 81\sin^2\phi\cos^2\phi}$$

$$= \sqrt{81\sin^2\phi(\sin^2\phi + \cos^2\phi)}$$

$$= \sqrt{81\sin^2\phi} = 9\sin\phi$$

Also, $x^2z + y^2z = (x^2+y^2)z = (\rho^2\sin^2\phi)\rho\cos\phi = 3^3\sin^2\phi\cos\phi$

$$\int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} 27\sin^2\phi\cos\phi \cdot 9\sin\phi\,d\theta\,d\phi$$

$$= 2\pi\int_{\frac{\pi}{2}}^{\pi} 243\sin^3\phi\cos\phi\,d\phi$$

$u = \sin\phi$, $du = \cos\phi\,d\phi$

$$= 486\pi\int_1^0 u^3\,du = 486\pi \cdot \frac{1}{4}u^4\Big|_1^0$$

$$= \boxed{-\frac{243}{2}\pi}$$