MA 261 - Lesson 3
Vector Functions

13.1 Vector Functions and Space Curves

A function is a rule that takes every mathematical object in a set called the domain and associates to it a unique mathematical object in a set called the codomain.

You are used to seeing functions where the domain and codomain are $\mathbb{R}$, i.e., the input and output are both real numbers.

In this lesson, we deal with vector functions which take real numbers as input and have vectors as output.

For example, $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ is a vector function. For any real number t, you get an output vector in $\mathbb{R}^3$.

e.g., $\vec{r}(0) = \langle \cos(0), \sin(0), 0 \rangle = \langle 1, 0, 0 \rangle$

$\vec{r}(\frac{\pi}{2}) = \langle \cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}), \frac{\pi}{2} \rangle = \langle 0, 1, \frac{\pi}{2} \rangle$

$\vec{r}(\pi) = \langle \cos(\pi), \sin(\pi), \pi \rangle = \langle -1, 0, \pi \rangle$

Domains of Vector Functions

In order for a vector function to make sense, all components must be defined for every t-value in the domain. So to find the domain, look at the domains for each component, and the vector function's domain is where the components' domains overlap.

Ex 1. Find the domain of $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

Domain of $\cos t$ is $\mathbb{R}$ (all real numbers)
Domain of $\sin t$ is $\mathbb{R}$ (all real numbers)
Domain of t is $\mathbb{R}$

So $\boxed{\vec{r}(t) \text{ has domain } \mathbb{R}}$

Ex 2. Find the domain of $\vec{r}(t) = \langle \frac{1}{t}, \ln(t+2), \sqrt{3-t} \rangle$

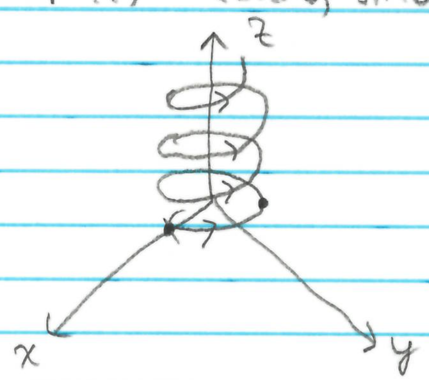
Domain of $\frac{1}{t}$ is all real numbers with $t \neq 0$
Domain of $\ln(t+2)$ is $t+2 > 0 \Leftrightarrow t > -2$
Domain of $\sqrt{3-t}$ is $3-t \geq 0 \Leftrightarrow t \leq 3$

Therefore, $\boxed{\vec{r}(t) \text{ has domain } (-2,0) \cup (0,3]}$

Space Curves

Given a vector function $\vec{r}(t)$, if we view t as a parameter, then placing the tail of $\vec{r}(t)$ at the origin, $\vec{r}(t)$ "traces out" a curve in space.

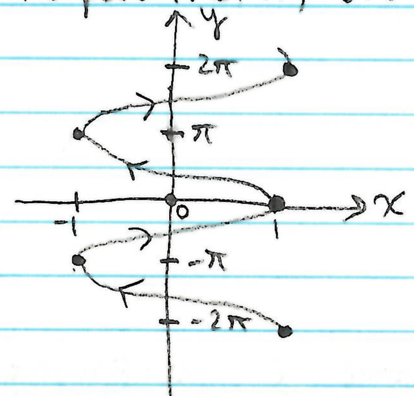
Consider $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$



We get a helix traveling upward as t increases.

Ex 3. Sketch the curve with vector equation $\vec{r}(t) = \langle \cos t, t \rangle$ and use arrows to indicate the direction as t increases.

Notice we get parametric equations $x = \cos t$, $y = t$. Eliminating the parameter, we get $x = \cos(y)$



Arrows must point in that direction since as t increases, y increases.

As the example above indicates, vector equations of space curves give rise to parametric equations of space curves, which we can use to find intersections.

Ex 4. At what point(s) does the helix $\vec{r}(t) = \langle \sin t, \cos t, t \rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

Helix has parametric equations $x = \sin t$, $y = \cos t$, $z = t$, so if the helix intersects the sphere, it must happen at t values where $(\sin t)^2 + (\cos t)^2 + (t)^2 = 5$

$$\iff 1 + t^2 = 5$$
$$\iff t^2 = 4$$
$$\iff t = \pm 2$$

So it happens at $\vec{r}(\pm 2)$ which are the points

$\boxed{(\sin(2), \cos(2), 2) \text{ and } (\sin(-2), \cos(-2), -2)}$

The intersection of two surfaces can be a space curve as well. To find a parameterization for the space curve of intersection, parametrize the surface with the simpler equation and plug into the equation of the other surface, and solve for the missing variable.

Ex 5. Find parametric equations for the curve of intersection of the parabolic cylinder $x = y^2$ and the bottom half of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{1} + \frac{z^2}{9} = 1$

Letting $y = t$, since the curve lies on the cylinder,

$$x = y^2 = (t)^2 = t^2$$

Since the curve also lies on the ellipsoid,

$$\frac{(t^2)^2}{4} + \frac{(t)^2}{1} + \frac{z^2}{9} = 1$$

$$\frac{z^2}{9} = 1 - \frac{t^4}{4} + t^2$$

$$\frac{z}{3} = \pm\sqrt{1 - \frac{t^4}{4} + t^2}$$

Only interested in bottom half, so $z = -3\sqrt{1 - \frac{t^4}{4} + t^2}$

So curve of intersection has parametric equations

$$\boxed{x = t^2, \quad y = t, \quad z = -3\sqrt{1 - \frac{t^4}{4} + t^2}}$$

Space Curves can represent trajectories of objects, as well.

Ex 6. Two particles travel along the space curves
$\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$, $\vec{r}_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle$.
Do the particles collide?

The particles collide if the is a t-value where their paths are at the same point, i.e., where

$$t = 1+2t, \quad t^2 = 1+6t, \quad t^3 = 1+14t$$

$$\Downarrow$$

$$t = -1$$

but $\quad (-1)^2 \neq 1+6(-1)$

So there is no t-value where they are at the same point.

13.2 Derivatives of Vector Functions

Given a vector function $\vec{r}(t) = \langle f(t), g(t), j(t) \rangle$, we define the derivative the same way as we do for real-valued functions,

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Notice $\displaystyle \lim_{h\to 0}\left(\frac{\langle f(t+h), g(t+h), j(t+h) \rangle - \langle f(t), g(t), j(t) \rangle}{h} \right)$

$$= \lim_{h\to 0} \left\langle \frac{f(t+h)-f(t)}{h}, \frac{g(t+h)-g(t)}{h}, \frac{j(t+h)-j(t)}{h} \right\rangle$$

$$= \left\langle \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}, \lim_{h\to 0} \frac{g(t+h)-g(t)}{h}, \lim_{h\to 0} \frac{j(t+h)-j(t)}{h} \right\rangle$$

$$= \langle f'(t), g'(t), j'(t) \rangle$$

Thus, we conclude that vector functions can be differentiated component-wise.
i.e.,

$$\boxed{\begin{array}{l}\text{If } \vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k},\\ \text{then } \frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\hat{\imath} + g'(t)\hat{\jmath} + h'(t)\hat{k}\end{array}}$$

The same reasoning holds for vector functions with any number of components.

Ex 7. Given $\vec{r}(t) = \langle \cos t, \ln(t+1), e^{-t}\rangle$, find $\vec{r}\,'(t)$.

$$\vec{r}\,'(t) = \langle \tfrac{d}{dt}\cos t, \tfrac{d}{dt}\ln(t+1), \tfrac{d}{dt}e^{-t}\rangle$$
$$= \boxed{\langle -\sin t, \tfrac{1}{t+1}, -e^{-t}\rangle}$$

Ex 8. Given $\vec{r}(t) = t^2\vec{a} + e^t\vec{b} + t\sin t\,\vec{c}$ for some constant vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, find $\vec{r}\,'(t)$.

Notice that if $f(t)$ is a function and $\vec{d}$ is a constant vector, then $\frac{d}{dt}(f(t)\vec{d}) = f'(t)\vec{d}$ since $\vec{d} = \langle d_1, d_2, d_3\rangle$ so $f(t)\vec{d} = \langle f(t)d_1, f(t)d_2, f(t)d_3\rangle$
So taking derivatives component-wise, we get
$\langle f'(t)d_1, f'(t)d_2, f'(t)d_3\rangle = f'(t)\langle d_1, d_2, d_3\rangle = f'(t)\vec{d}$.

So $\boxed{\vec{r}\,'(t) = 2t\vec{a} + e^t\vec{b} + (t\cos t + \sin t)\vec{c}}$

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Just like derivatives of real-valued functions, $\vec{r}\,'(a)$ is a vector that lies tangent to the space curve $\vec{r}(t)$ at $\vec{r}(a)$.

This can be seen from the definition of $\vec{r}\,'(t)$ as $\lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$, which is a secant vector of the space curve which tends to the tangent vector as $h \to 0$.

Hence, $\vec{r}\,'(t)$ is called the <u>tangent vector</u> to the curve defined by $\vec{r}(t)$

<u>Ex 9</u>. Sketch the plane curve given by $\vec{r}(t) = \langle t^2, t^3 \rangle$, then sketch the position vector $\vec{r}(1)$ from the origin the curve and the tangent vector $\vec{r}\,'(1)$ with tail at $\vec{r}(1)$.

$\vec{r}\,'(t) = \langle 2t, 3t^2 \rangle$

$\vec{r}(1) = \langle 1^2, 1^3 \rangle = \langle 1, 1 \rangle$, $\vec{r}\,'(1) = \langle 2(1), 3(1)^2 \rangle = \langle 2, 3 \rangle$

To sketch $\vec{r}(t)$, try some values of t

t	x	y
-3	9	-27
-2	4	-8
-1	1	-1
0	0	0
1	1	1
2	4	8
3	9	27