MA 261 - Lesson 4

Arc Length and Curvature (13.3)

From Calc 2, we know that given a plane curve with parametric equations $x = f(t)$, $y = g(t)$, the arclength of the curve from $t=a$ to $t=b$ is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

which can be seen by taking a Riemann sum of tangent line segments as the step size approaches 0.

By the same argument, if we have a vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, the arc length of the space curve from $t=a$ to $t=b$ is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

Notice that the integrand is simply $|\vec{r}\,'(t)|$, the length of the tangent vector. Thus, we get

$$\boxed{L = \int_a^b |\vec{r}\,'(t)|\, dt}$$

Ex 1. Find the length of the curve $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + \ln(\cos t)\,\hat{k}$ for $0 \le t \le \frac{\pi}{4}$

$$\vec{r}\,'(t) = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \frac{1}{\cos t}\cdot -\sin t\,\hat{k} = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} - \tan t\,\hat{k}$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = |\sec t|$$

On $0 \le t \le \frac{\pi}{4}$, $\sec t \ge 0$, so $|\vec{r}\,'(t)| = \sec t$

(Ex 1 continued)

So $L = \int_0^{\pi/4} \sec t \, dt$

$= \int_0^{\frac{\pi}{4}} \sec t \cdot \frac{\sec t + \tan t}{\sec t + \tan t} \, dt$

$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 t + \sec t \tan t}{\sec t + \tan t} \, dt$

Let $u = \sec t + \tan t$

$du = (\sec t \tan t + \sec^2 t) \, dt$

$u(\frac{\pi}{4}) = \sec(\frac{\pi}{4}) + \tan(\frac{\pi}{4}) = \sqrt{2} + 1$

$u(0) = \sec(0) + \tan(0) = 1$

$\int_1^{\sqrt{2}+1} \frac{1}{u} \, du$

$= \ln|u| \Big|_1^{\sqrt{2}+1} = \ln(\sqrt{2}+1) - \underbrace{\ln(1)}_{0}$

$= \boxed{\ln(\sqrt{2}+1)}$

Ex 2. Find the length of the curve $\vec{r}(t) = \langle \frac{1}{2}t^2, \sqrt{2}t, \ln t \rangle$ from $1 \le t \le e$.

$\vec{r}\,'(t) = \langle t, \sqrt{2}, \frac{1}{t} \rangle$

$|\vec{r}\,'(t)| = \sqrt{t^2 + 2 + \frac{1}{t^2}} = \sqrt{(t + \frac{1}{t})^2} = |t + \frac{1}{t}| = t + \frac{1}{t}$

since $t + \frac{1}{t} \ge 0$ when $1 \le t \le e$

$L = \int_1^e (t + \frac{1}{t}) \, dt = (\frac{1}{2}t^2 + \ln t) \Big|_1^e$

$= (\frac{1}{2}e^2 + \ln(e)) - (\frac{1}{2} + \ln(1))$

$= \frac{1}{2}e^2 + 1 - \frac{1}{2} - 0$

$= \frac{1}{2}e^2 + \frac{1}{2}$

$= \boxed{\frac{e^2+1}{2}}$

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If we want to find the arc length from a fixed point to multiple points, it might be useful to establish an arclength function.

Given a vector function $\vec{r}(t)$, the arclength function from point $\vec{r}(a)$ in the direction of increasing t is

$$S(t) = \int_a^t |\vec{r}\,'(u)|\,du$$

Ex 3. Find the arclength function for the curve measured from $P(0,1,\sqrt{2})$ in direction of increasing t for $\vec{r}(t) = \langle e^t \sin t, e^t \cos t, \sqrt{2}\, e^t \rangle$.

$P(0,1,\sqrt{2})$ occurs when $0 = e^t \sin t$, $1 = e^t \cos t$, $\sqrt{2} = \sqrt{2}\, e^t$ which is when $t = 0$.

$$\vec{r}\,'(t) = \langle e^t \cos t + e^t \sin t, -e^t \sin t + e^t \cos t, \sqrt{2}\, e^t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{[e^t(\cos t + \sin t)]^2 + [e^t(\cos t - \sin t)]^2 + [\sqrt{2}\, e^t]^2}$$

$$= \sqrt{e^{2t}(\cos^2 t + 2\cos t \sin t + \sin^2 t) + e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t) + 2e^{2t}}$$

$$= \sqrt{4e^{2t}} = 2e^t$$

So $S(t) = \int_0^t 2e^u\,du = 2e^u \Big|_0^t = \boxed{2e^t - 2}$

Any space curve can be parameterized in many ways, but one that is less arbitrary is to parameterize with respect to arclength from a fixed point, since it is based on the intrinsic properties of the curve itself.

To do this, we can solve the arclength function for t, making t a function of s, and then substituting into the vector function.

Ex 4. In Ex 3, we saw that the arclength function for $\vec{r}(t) = \langle e^t \sin t, e^t \cos t, \sqrt{2}\, e^t \rangle$ from $P(0, 1, \sqrt{2})$ in the direction of increasing t is $s(t) = 2e^t - 2$. Parameterize the space curve with respect to arclength, and determine the point 4 units along the curve (in the direction of increasing t) from P.

$$s = 2e^t - 2 \iff s + 2 = 2e^t \iff \frac{s}{2} + 1 = e^t$$
$$\iff \ln\left(\frac{s}{2}+1\right) = t$$

So
$$\vec{r}(s) = \left\langle e^{\ln(\frac{s}{2}+1)} \sin\left(\ln\left(\tfrac{s}{2}+1\right)\right), e^{\ln(\frac{s}{2}+1)} \cos\left(\ln\left(\tfrac{s}{2}+1\right)\right), \sqrt{2}\, e^{\ln(\frac{s}{2}+1)} \right\rangle$$
$$= \left\langle \left(\tfrac{s}{2}+1\right) \sin\left(\ln\left(\tfrac{s}{2}+1\right)\right), \left(\tfrac{s}{2}+1\right) \cos\left(\ln\left(\tfrac{s}{2}+1\right)\right), \sqrt{2}\left(\tfrac{s}{2}+1\right) \right\rangle$$

$$\vec{r}(4) = \langle 3\sin(\ln(3)), 3\cos(\ln(3)), 3\sqrt{2} \rangle$$

so the point 4 units along the curve from P (in direction of increasing t) is $\boxed{(3\sin(\ln(3)), 3\cos(\ln(3)), 3\sqrt{2})}$

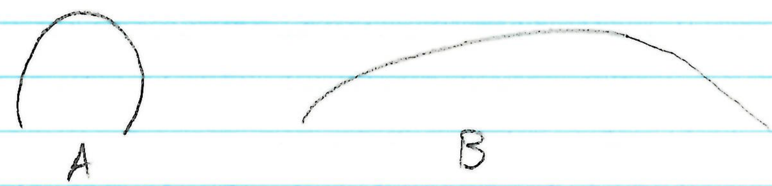
Curvature

We say that a curve is smooth if it has no drastic changes. In other words, if the curve is changing direction consistent with its tangent vector.

Therefore, we say a curve $\vec{r}(t)$ is **smooth** if $\vec{r}\,'(t)$ exists and $\vec{r}\,'(t) \neq \vec{0}$.
(The $\vec{0}$ has no defined direction)

For a smooth curve, it makes sense to ask just how curvy that curve is. So we want to develop a notion of curvature.

Consider the curves



We want to say that A has a greater curvature than B. How can we do this?

Notice that the tangent vector to A changes direction more quickly than the tangent vector to B. So curvature should be based on the rate of change of the direction of the tangent vector.

Since we only care about how the direction of the tangent vector changes, we look at the unit tangent vector denoted $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

So in order to get a number representing how quickly $\vec{T}(t)$ changes, we would naturally look at $|\vec{T}\,'(t)|$.

Since $|\vec{T}(t)| = 1$, $\vec{T}\,'(t)$ is actually orthogonal to $\vec{T}(t)$.
This is because $\vec{T} \cdot \vec{T} = |\vec{T}|^2 = 1$; hence $\frac{d}{dt}[\vec{T} \cdot \vec{T}] = 0$.
On the other hand, $\frac{d}{dt}[\vec{T} \cdot \vec{T}] = \vec{T} \cdot \vec{T}\,' + \vec{T} \cdot \vec{T}\,' = 2\vec{T} \cdot \vec{T}\,'$.
Hence $2\vec{T} \cdot \vec{T}\,' = 0$ so $\vec{T} \cdot \vec{T}\,' = 0$.
As such $\vec{T}\,'(t)$ is a vector normal to the curve $\vec{r}(t)$, and we call $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$ the unit normal vector.

Back to the idea of curvature, we think a good notion for the curvature of a curve $\vec{r}(t)$ should be $|\vec{T}'(t)|$.
But there's a problem!
The plane circle $x^2+y^2=1$ can be given the two different parameterizations

$$\vec{r}_1(t) = \langle \cos t, \sin t \rangle \quad \text{or} \quad \vec{r}_2(t) = \langle \cos(2t), \sin(2t) \rangle$$

$$\vec{r}_1{}'(t) = \langle -\sin t, \cos t \rangle \qquad \vec{r}_2{}'(t) = \langle -2\sin(2t), 2\cos(2t) \rangle$$

$$|\vec{r}_1{}'(t)| = 1 \qquad |\vec{r}_2{}'(t)| = 2$$

$$\vec{T}_1(t) = \langle -\sin t, \cos t \rangle \qquad \vec{T}_2(t) = \langle -\sin(2t), \cos(2t) \rangle$$

$$\vec{T}_1{}'(t) = \langle -\cos t, -\sin t \rangle \qquad \vec{T}_2{}'(t) = \langle -2\cos(2t), -2\sin(2t) \rangle$$

$$|\vec{T}_1{}'(t)| = 1 \qquad |\vec{T}_2{}'(t)| = 2$$

Even though the two parameterizations $\vec{r}_1$ and $\vec{r}_2$ represent the same curve, they give different values for what we thought curvature should be!

In order to give a unique value for curvature which is based on intrinsic properties of the curve rather than arbitrary parameterizations, we parameterize with respect to arclength, and this makes sense because looking at curves A and B, when talking about "how quickly" the tangent direction changes, we were really thinking about moving distances along the curves anyway.

So the curvature K of a curve is defined to be $$K = \left| \frac{d\vec{T}}{ds} \right|$$

Ex 5. In Ex 4, we saw that a space curve $\vec{r}(t) = \langle e^t \sin t, e^t \cos t, \sqrt{2}e^t \rangle$ could be parameterized in terms of arclength from $P(0,1,\sqrt{2})$ as
$\vec{r}(s) = \langle (\frac{s}{2}+1)\sin(\ln(\frac{s}{2}+1)), (\frac{s}{2}+1)\cos(\ln(\frac{s}{2}+1)), \sqrt{2}(\frac{s}{2}+1) \rangle$.
Find the curvature at P.

$$\vec{r}'(s) = \langle (\tfrac{s}{2}+1)\cos(\ln(\tfrac{s}{2}+1))\frac{1}{s/2+1}\cdot\frac{1}{2} + \frac{1}{2}\sin(\ln(\tfrac{s}{2}+1)),$$
$$-(\tfrac{s}{2}+1)\sin(\ln(\tfrac{s}{2}+1))\frac{1}{s/2+1}\cdot\frac{1}{2} + \frac{1}{2}\cos(\ln(\tfrac{s}{2}+1)), \frac{\sqrt{2}}{2} \rangle$$

$$|\vec{r}'(s)| = \sqrt{\frac{1}{4}\cos^2(\ln(\tfrac{s}{2}+1)) + \frac{1}{2}\cos(\ln(\tfrac{s}{2}+1))\sin(\ln(\tfrac{s}{2}+1)) + \frac{1}{4}\sin^2(\ln(\tfrac{s}{2}+1)) + \frac{1}{4}\sin^2(\ln(\tfrac{s}{2}+1)) - \frac{1}{2}\cos(\ln(\tfrac{s}{2}+1))\sin(\ln(\tfrac{s}{2}+1)) + \frac{1}{4}\cos^2(\ln(\tfrac{s}{2}+1)) + \frac{1}{2}}$$
$$= \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{2}} = \sqrt{1} = 1$$

so $\vec{T}(s) = \langle \frac{1}{2}\cos(\ln(\frac{s}{2}+1)) + \frac{1}{2}\sin(\ln(\frac{s}{2}+1)), \frac{1}{2}\cos(\ln(\frac{s}{2}+1)) - \frac{1}{2}\sin(\ln(\frac{s}{2}+1)), \frac{\sqrt{2}}{2} \rangle$

$$\vec{T}'(s) = \langle -\frac{1}{2}\sin(\ln(\tfrac{s}{2}+1))\frac{1}{s/2+1}\cdot\frac{1}{2} + \frac{1}{2}\cos(\ln(\tfrac{s}{2}+1))\cdot\frac{1}{s/2+1}\cdot\frac{1}{2},$$
$$-\frac{1}{2}\sin(\ln(\tfrac{s}{2}+1))\frac{1}{s/2+1}\cdot\frac{1}{2} - \frac{1}{2}\cos(\ln(\tfrac{s}{2}+1))\frac{1}{s/2+1}\cdot\frac{1}{2}, 0 \rangle$$

We care about $|\vec{T}'(0)|$

$$\vec{T}'(0) = \langle -\frac{1}{2}\sin(0)\cdot 1\cdot\frac{1}{2} + \frac{1}{2}\cos(0)\cdot\frac{1}{1}\cdot\frac{1}{2}, -\frac{1}{2}\sin(0)\cdot\frac{1}{1}\cdot\frac{1}{2} - \frac{1}{2}\cos(0)\cdot\frac{1}{1}\cdot\frac{1}{2}, 0 \rangle$$
$$= \langle \frac{1}{4}, -\frac{1}{4}, 0 \rangle$$

$$|\vec{T}'(0)| = \sqrt{\frac{1}{16}+\frac{1}{16}+0} = \sqrt{\frac{1}{8}} = \boxed{\frac{1}{2\sqrt{2}} = K}$$

You may think that this is a tedious process, and it is! Even though we need to use arclength to get curvature in theory, the chain rule saves the day and allows us to skip arclength!

By the chain rule, we know $\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$

Thus, $\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} = \frac{d\vec{T}/dt}{\frac{d}{dt}\left(\int_a^t |\vec{r}\,'(u)|\, du\right)} = \frac{d\vec{T}/dt}{|\vec{r}\,'(t)|}$

Hence, $K = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$

Your book calls this equation 9:

$$\boxed{K(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}}$$

Thus, we can more easily find curvature using any parameterization we like.

Ex 6. For $\vec{r}(t) = \langle 4\cos t, 4\sin t \rangle$, find $\vec{T}(0)$, $\vec{N}(0)$, $K(0)$.

$\vec{r}\,'(t) = \langle -4\sin t, 4\cos t \rangle$

$|\vec{r}\,'(t)| = \sqrt{16\sin^2 t + 16\cos^2 t} = \sqrt{16} = 4$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin t, \cos t \rangle$

$\vec{T}(0) = \langle -\sin 0, \cos 0 \rangle = \boxed{\langle 0, 1 \rangle}$

$\vec{T}\,'(t) = \langle -\cos t, -\sin t \rangle$

$|\vec{T}\,'(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \langle -\cos t, -\sin t \rangle$

$\vec{N}(0) = \langle -\cos 0, -\sin 0 \rangle = \boxed{\langle -1, 0 \rangle}$

$K(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{1}{4}$, so $\boxed{K(0) = \frac{1}{4}}$

In general, $K(t)$ can be a function of t.