Math 266, Practice Midterm 1 solutions

1. *Which of the following differential equations is separable?*

   (a) $\frac{dy}{dx} = \frac{x^2+xy+y^2}{2y^2}$

   (b) $-2\ln(x) + 3yy' = 0$

   (c) $(y + 3\cos(y))\,dx + (x - 3x\sin(y))\,dy = 0$

   (d) $y' + 3\ln(x)y = 2x^2 + e^x$

   The correct answer is (b). Notice that we can rewrite it as:

   $$3y\,dy = 2\ln(x)\,dx.$$

   The other ones should also look familiar. (a) is homogeneous, (c) is exact, and (d) is linear – so you know how to solve all four of these.

2. *Which of the following functions is the integrating factor for the equation $ty' = 2y + t^3$?*

   (a) e^{2t}

   (b) e^{-2t}

   (c) $2/t$

   (d) $2t$

   I actually messed this up! Thanks to all those who emailed me about it. To find the integrating factor, we write the equation as

   $$y' - (2/t)y = t^2$$

   and compute

   $$\mu(t) = \exp\left(\int (-2/t)\,dt\right) = \exp(-2\ln(t)) = t^{-2}.$$

3. *A large tank is full of a corrosive fluid, which is leaking out of a small circular hole at a rate proportional to the fourth power of the hole's radius. As the fluid leaks, it corrodes the edges of the hole and enlarges it. The area of the hole increases at a rate proportional to the amount of fluid that has leaked out. Write (don't solve!) a differential equation that describes this situation.*

   This was also poorly written. What I meant to say was that *the area of the hole increases at a rate proportional to the flow rate of the fluid.* Let F be the flow rate of the fluid, and r the radius of the hole – both are functions of time. We have

   $$F = \alpha r^4,$$

   where α is a constant. Let A be the area of the hole – we have $A = \pi r^2$. The rate of change of the hole's area is

   $$dA/dt = \beta F = \alpha\beta r^4.$$

   Since $A = \pi r^2$, the left-hand side is

   $$d(\pi r^2)/dt = 2\pi r \cdot \frac{dr}{dt}.$$

Thus, we have a differential equation for r,

$$\frac{dr}{dt} = \frac{\alpha\beta}{2\pi} r^3 = kr^3$$

where k is a constant.

But there's another way of reading the problem as written: *the rate of increase of the area of the hole is proportional to the amount of fluid that has leaked out already.* We can model this mathematically, too, although it's a little more complicated. The amount of fluid that has leaked out at time t is

$$\int_0^t F\,dt$$

so we have

$$dA/dt = \beta \int_0^t F\,dt = \int_0^t \alpha\beta r^4\,dt.$$

This isn't a differential equation, but we can get a *second-order* differential equation by differentiating again and using the fundamental theorem of calculus:

$$\frac{d^2A}{dt^2} = \alpha r^4.$$

To get a differential equation for r, we have to compute the left-hand side. We saw that

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Differentiating again and using the product rule gives

$$\frac{d^2A}{dt^2} = 2\pi \left(\frac{dr}{dt}\right)^2 + 2\pi r \frac{d^2r}{dt^2}.$$

So we get a second-order nonlinear equation:

$$r \cdot r'' + (r')^2 = kr^3$$

where $k = (\alpha\beta)/2\pi$ is a constant, as before.

4. *Solve the following initial value problem, and give the maximum interval on which the solution is valid.*

$$ty' + 4y = t^{-2}e^t, \quad y(1) = 2.$$

This is a first-order linear equation. Begin by dividing by t to get the coefficient of y' to be 1:

$$y' + (4/t)y = t^{-3}e^t.$$

We now multiply by the integrating factor

$$\mu(t) = \exp\left(\int (4/t)\,dt\right) = \exp(4\ln(t)) = t^4.$$

This gives us

$$(t^4y)' = t^4y' + 4t^3y = te^t.$$

Integrate:

$$t^4y = \int te^t\,dt.$$

The right-hand side can be integrated by parts, with $u = t$ and $dv = e^t\,dt$. We get

$$t^4 y = te^t - \int e^t = te^t - e^t + C.$$

Thus,

$$y = t^{-3}e^t - t^{-4}e^t + Ct^{-4}.$$

When $t = 1$,

$$y = e - e + C = C,$$

so $C = 2$ for this initial condition. The solution is

$$y = t^{-3}e^t - t^{-4}e^t + 2t^{-4}.$$

Notice that this function is discontinuous just at $t = 0$. This is confirmed by the differential equation: the functions $4/t$ and $t^{-3}e^t$ are just discontinuous at $t = 0$. Since the differential equation is linear, we know that solutions to initial value problems exist and are unique in any interval on which these two functions are continuous. Thus, the given solution is valid in the interval $(0, \infty)$.

5. *Solve the following initial value problem, and give the maximum interval on which the solution is valid.*

$$y' = \frac{4x}{1+2y}, y(1) = -1.$$

This equation is not linear, but it is separable. We separate the variables to get

$$(1 + 2y)\,dy = 4x\,dx,$$

which integrates to

$$y^2 + y = 2x^2 + C.$$

For the given initial condition, we have

$$1 - 1 = 2 + C,$$

so $C = -2$.

To get y as a function of x, rewrite the equation as

$$y^2 + y + (-2x^2 + 2) = 0$$

and solve for y using the quadratic formula:

$$y = \frac{-1 \pm \sqrt{1 + 8x^2 - 8}}{2} = \frac{-1 \pm \sqrt{8x^2 - 7}}{2}.$$

So there are two possible functions of x that solve the equation. To see which one is right, look at the initial condition again: we have to pick the minus sign here. Thus, the answer is

$$y = \frac{-1 - \sqrt{8x^2 - 7}}{2}.$$

As you can see, this stops being defined (i.e., real) when $8x^2 < 7$, or when

$$-\sqrt{7/8} < x < \sqrt{7/8}.$$

Note also that when $x^2 = 7/8$, $y = -1/2$, which is the only singularity of y' appearing in the original differential equation. Since there is an initial condition given at $x = 1$, the solution we care about is the one with positive x. So the solution we care about is defined on the interval $[\sqrt{7/8}, \infty)$. (It is also fine to write $(\sqrt{7/8}, \infty)$. When $x = \sqrt{7/8}$, y is defined by y' is not, so the differential equation could be considered not to hold at this point.)

6. *Find all linear solutions (functions of the form $y = mx + b$) to the differential equation*

$$\frac{dy}{dx} = \frac{x^2 - 2y^2}{2xy}.$$

You can solve this by plugging in $mx + b$ for y and solving for m and b. The problem is made slightly simpler if you first notice that the equation is homogeneous, and can be written as

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{y}{x}\right)^{-1} - \frac{y}{x}.$$

Since y' is a function of y/x, it is constant along every line through the origin. On the other hand, for $y = mx + b$ to solve the differential equation, $y' = m$ must be constant along this line. The only way this can happen is if the line goes through the origin, i. e., if $b = 0$. You would figure this out if you solved the problem directly, as well.

If $y = mx$, then $y' = m$ and the equation reduces to

$$m = \frac{1}{2m} - m$$

or

$$4m^2 = 1.$$

The solutions are $m = \pm 1/2$. These are the lines $y = \pm x/2$.

7. *An Indonesian island can support a population of up to 600 cassowaries. The birds only reproduce enough to grow the population if there are more than 100 of them – if there are any fewer, the population begins to die out.*

   (a) *Write a differential equation that describes the population of cassowaries on the island. Define any symbols you introduce.*

   (b) *Sketch a direction field for your equation.*

One possible answer is a logistic equation with a threshold. We expect the population, P, to grow logistically if $P > 100$, up to the carrying capacity $P = 600$. Maybe you remember this formula from the book, but let's say you don't. How would you come up with it? Well, the setup tells us that

$$P' \begin{cases} < 0 & P < 100 \text{ (the population dies out)} \\ > 0 & 100 < P < 600 \text{ (the population grows)} \\ < 0 & P > 600 \text{ (too many for the environment to support)} \end{cases}$$

So we should have $P' = 0$ at $P = 100$ and 600. We probably also want $P' = 0$ when $P = 0$ (if there are no cassowaries, they can't reproduce), and we probably don't want any other zeros of P'. Finally, we expect the differential equation to be autonomous – P' should depend on P but not on time.

So, what's a function of P with these properties? Well, the simplest *polynomial* function is

$$P' = -rP(P - 100)(P - 600),$$

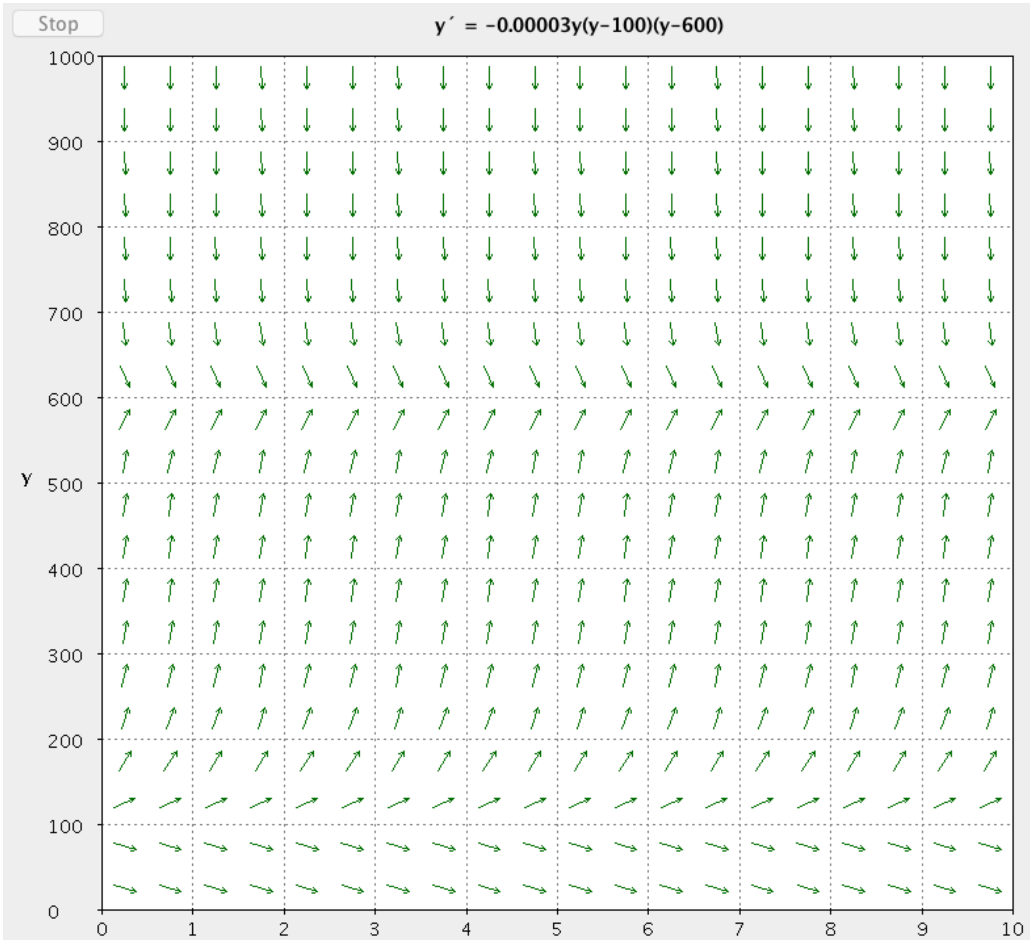
where r is a positive constant. Note that we need a negative number $-r$ so that the function has the right signs. This is basically the logistic-equation-with-a-threshold discussed in class and in your book.

But any function with these properties will work! For example, you could have tried to modify the Gompertz equation, and come up with something like

$$P' = -\ln(P/100)\ln(P/600).$$

This goes to $-\infty$ as $P \to 0$ (so a Gompertz population would probably go extinct quicker than a logistic one), but otherwise has the right behavior.

Here's a direction field from dfield:



To get points on a question like this, your graph would need to be clear and properly labelled, the slopes shouldn't depend on time, and the slopes should be approximately correct (and in particular, have the right signs).

8. *Consider the differential equation*

$$xy^2 + Ax^2y + (x^3 + yx^2)\frac{dy}{dx} = 0.$$

Find the value of A that makes the equation exact. Solve the equation for this value of A.

For any value of A, the functions appearing in the equation are continuous and have continuous partial derivatives everywhere. The criterion for exactness for such an equation of the form

$$M + N\frac{dy}{dx} = 0$$

is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

In this case, this means

$$2xy + Ax^2 = 3x^2 + 2xy,$$

so that $A = 3$.

We now want to find a potential function ψ such that

$$\frac{\partial \psi}{\partial x} = xy^2 + 3x^2 y, \quad \frac{\partial \psi}{\partial y} = x^3 + x^2 y.$$

Partially integrating should lead you to

$$\psi(x, y) = \frac{x^2 y^2}{2} + x^3 y.$$

Thus, the general solution to the equation is

$$\frac{x^2 y^2}{2} + x^3 y = C.$$

Since this is quadratic in y, we could solve for y as a function of x. However, this isn't necessary for this type of problem.

9. *Use the substitution $v = y^3$ to find the general solution to the differential equation*

$$y' = 5y + e^{-2x} y^{-2}.$$

If $v = y^3$, then

$$v' = 3y^2 y',$$

which is to say that

$$y' = v'/3y^2.$$

Plugging this in for y', we get

$$v'/3y^2 = 5y + e^{-2x} y^{-2}$$

or

$$v' = 15y^3 + 3e^{-2x} = 15v + 3e^{-2x}.$$

This is a linear equation for v as a function of x, and can be written as

$$v' - 15v = 3e^{-2x}.$$

The integrating factor is $\mu(x) = e^{-15x}$. We get

$$e^{-15x} v = \int 3e^{-17x}\, dx = \frac{-3}{17} e^{-17x} + C.$$

Thus,

$$v = \frac{-3}{17} e^{-2x} + Ce^{15x}.$$

Finally, we need a solution in terms of y, but $y = v^{1/3}$. So

$$y = \left(\frac{-3}{17} e^{-2x} + Ce^{15x} \right)^{1/3}.$$