AN EXAMPLE OF REDUCTION OF ORDER

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Consider the differential equation

$$(t-1)y'' - ty' + y = 0. \tag{1}$$

This doesn't fall into any of the nice classes of equation that we've studied. Nevertheless, one solution to this is

$$y_1 = e^t.$$

You might be able to guess this, for example, by noting that the coefficients add up to $(t-1) - t + 1 = 0$, so that if $y'' = y' = y$, then the solution is zero; and the only way (not counting scalar multiples) to make $y'' = y' = y$ is to put $y = e^t$. Or you might have found it by trying different functions and seeing what got things to cancel out.

Let $y_2 = y_1 v$. Then

$$y_2' = y_1' v + y_1 v',$$
$$y_2'' = y_1'' v + 2y_1' v' + y_1 v''.$$

Substituting this into the original equation gives

$$(t-1)(y_1'' v + 2y_1' v' + y_1 v'') - t(y_1' v + y_1 v') + y_1 v = 0$$

and we can rearrange the terms to get

$$(t-1)y_1 v'' + \left[2(t-1)y_1' - y_1 t\right] v' + \left[(t-1)y_1'' - ty_1' + y_1\right] v = 0.$$

The coefficient of v is just what you get from plugging y_1 into (1). Since y_1 was a solution to the equation, this coefficient is zero. We get

$$(t-1)e^t v'' + \left[2(t-1)e^t - te^t\right] v' = 0.$$

We can divide by e^t, since it's nonzero everywhere.

$$(t-1)v'' + (t-2)v' = 0.$$

(If you're in my 3:30 class – this last step is where I made a mistake. I got $2t-3$ rather than $t-2$ for the coefficient of v'.)

Now let $w = v'$. We have a first-order equation for w:

$$(t-1)w' + (t-2)w = 0.$$

This is separable. In fact, you'll always get a separable equation at this point – do you see why? We separate the variables to get

$$\int \frac{1}{w}\, dw = \int -\frac{t-2}{t-1}\, dt = \int \left(-1 + \frac{1}{t-1}\right) dt.$$

Integrating gives

$$\ln|w| = -t + \ln|t-1| + C$$

or

$$|w| = e^{-t} \cdot |t-1| \cdot e^C.$$

Writing $A = \pm e^C$ lets us remove the absolute value signs.

$$w = A(t-1)e^{-t}.$$

We now have to integrate w to get v.

$$v = \int A(t-1)e^{-t}\,dt = \int Ate^{-t}\,dt - \int Ae^{-t}\,dt. \tag{2}$$

The first integral is done by parts. Let $f = At$ and $dg = e^{-t}\,dt$, so that $df = A\,dt$ and $g = -e^{-t}$. Then

$$\int Ate^{-t}\,dt = \int f\,dg = fg - \int g\,df = -Ate^{-t} + \int Ae^{-t}\,dt.$$

Putting this back into (2), the two remaining integrals cancel out except for a constant. So

$$v = -Ate^{-t} + C.$$

Finally,

$$y_2 = y_1 v = -At + Ce^t.$$

Since we never picked A and C, this gave us the general solution to (1). There's a simple reason for this: *any* solution can be written as $y = y_1 v$ – just define $v = y/y_1$! A fundamental set of solutions is given by $\{e^t, t\}$.

You should check for yourself that t actually solves (1). Since this is also a very simple function, you might have found it first in the playing-around-with-the-equation stage – and then you could use reduction of order to find the other fundamental solution, e^t.