THE METHOD OF UNDETERMINED COEFFICIENTS FOR NONHOMOGENEOUS LINEAR SYSTEMS

Consider the system of differential equations

\[ \mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^{2t} \\ -2e^t \end{pmatrix}. \]

By way of analogy, I’m going to call the function \( \mathbf{g} \), or other functions in the same position, a “forcing function”, even though this isn’t necessarily a spring problem.

Let’s solve this using the method of undetermined coefficients. The general solution is of the form

\[ \mathbf{x} = \mathbf{x}_C + \mathbf{x}_P \]

where \( \mathbf{x}_C \) is the general solution to the associated homogeneous equation, and \( \mathbf{x}_P \) is a particular solution to the equation (1).

So we start by solving the associated homogeneous equation

\[ \mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x}. \]

The characteristic polynomial is

\[ \det(A - \lambda I) = (1 - \lambda)(-2 - \lambda) - 1 \cdot 4 = \lambda^2 + \lambda - 6, \]

which factors as \((\lambda + 3)(\lambda - 2)\). So the eigenvalues are \( \lambda = -3 \) and \( 2 \).

An eigenvector \( \xi \) for \( \lambda = -3 \) will satisfy

\[ (A + 3I)\xi = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\xi = 0. \]

One such vector is \( \begin{pmatrix} 1 \\ -4 \end{pmatrix} \).

An eigenvector for \( \lambda = 2 \) will satisfy

\[ (A - 2I)\xi = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} = 0. \]

One such vector is \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).

So the general solution to (2) is

\[ \mathbf{x}_C = C_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}. \]

We now find a particular solution to (1). First, let’s rewrite (1) as

\[ \mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^t. \]

This can be split into two nonhomogeneous equations with the same AHE:

\[ \mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \]

\[ \mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^t \]

Let \( \mathbf{x}_P^{(1)} \) be a particular solution to (4), and let \( \mathbf{x}_P^{(2)} \) be a particular solution to (5). Then \( \mathbf{x}_P^{(1)} + \mathbf{x}_P^{(2)} \) is a particular solution to (3). (Do you see why? Try substituting this into (3) and using what you know about \( \mathbf{x}_P^{(1)} \) and \( \mathbf{x}_P^{(2)} \). This is similar to the reason why the sum of two solutions to a linear homogeneous equation is also a solution to the same equation.) So the general solution to (1) can be written in the form

\[ \mathbf{x} = \mathbf{x}_C + \mathbf{x}_P^{(1)} + \mathbf{x}_P^{(2)}. \]
Next, we find \( x_p^{(2)} \). The forcing function in (5) is a multiple of \( e^t \) (by a vector!), so we should try the function with undetermined (vector-valued!) coefficients,

\[ x = ae^t = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t. \]

Let’s substitute this into (5). We get

\[ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{pmatrix} a_1 + a_2 \\ 4a_1 - 2a_2 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^t. \]

(You might be tempted to solve for \( t \) here, but that’s the wrong move! The two sides are equal as functions of \( t \), so we can just compare the coefficients of \( e^t \) and solve for \( a_1 \) and \( a_2 \).) Comparing the coordinates, we get the system of equations

\[ a_1 = a_1 + a_2, \]
\[ a_2 = 4a_1 - 2a_2 + 2. \]

(Be sure you understand this step, since it’s the sort of vector manipulation you’ll need to do quite a bit.) The solution is \( a_2 = 0, a_1 = 1/2 \). So our particular solution to (5) is

\[ x_p^{(2)} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^t. \]

(This illustrates what you should be doing in general. If the forcing function is a vector times \( e^{rt} \), try an unknown vector times \( e^{rt} \). If it’s a vector times \( \sin(rt) \) or \( \cos(rt) \), try an unknown vector times \( \sin(rt) \) plus an unknown vector times \( \cos(rt) \). If it’s a vector times a polynomial of degree \( d \), try a polynomial of degree \( d \) with unknown vector coefficients. If it’s a product of these, try a product of the appropriate things, just as you would in the single-equation case. As always, don’t be afraid to try something and get it wrong – you can usually see what you missed after doing the calculation, and correct accordingly.)

Finally, we have to find \( x_p^{(1)} \). Of course, this is a little subtler since the forcing function in (4) is a multiple of \( e^{2t} \), and 2 is a root of the characteristic polynomial. I’ll do multiple attempts to show you what happens.

**First attempt:** We try

\[ x = be^{2t} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{2t}. \]

Plugging this into (4), we get

\[ \begin{pmatrix} 2b_1 \\ 2b_2 \end{pmatrix} e^{2t} = \begin{pmatrix} b_1 + b_2 \\ 4b_1 - 2b_2 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}. \]

Comparing the coefficients of \( e^{2t} \) on both sides, we get the system of equations

\[ 2b_1 = b_1 + b_2 + 1, \]
\[ 2b_2 = 4b_1 - 2b_2. \]

It’s not too hard to see that this system has no solutions. So this choice of \( x \) didn’t work.

Here’s one way of thinking about what just happened. For certain values of \( b \) – namely, for \( b \) a scalar multiple of the eigenvector \( (1) - b e^{2t} \) is a solution to the associated homogeneous equation. Substituting any of these functions into \( x' - Ax \) gives zero. So substituting an arbitrary function \( be^{2t} \) into \( x' - Ax \) gives just a one-dimensional space of outputs, where we sort of need a two-dimensional one. You can check that if \( x = be^{2t} \), then \( x' - Ax \) is a scalar multiple of \( (1)e^{2t} \). If our forcing function was also a scalar multiple of \( (1)e^{2t} \), we’d be able to solve for \( x \), but in general, this is too little space to move in.

**Second attempt:** Inspired by the single-variable case, let

\[ x = be^{2t} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^{2t}. \]

Substituting this into (4), we obtain

\[ \begin{pmatrix} 2b_1 \\ 2b_2 \end{pmatrix} te^{2t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{2t} = \begin{pmatrix} b_1 + b_2 \\ 4b_1 - 2b_2 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}. \]
Comparing the coefficients of $te^{2t}$, we get
\[
2b_1 = b_1 + b_2, \\
2b_2 = 4b_1 - 2b_2.
\]
These equations are satisfied whenever $b_1 = b_2$. However, comparing the coefficients of $e^{2t}$, we also must have $b_1 = 1$ and $b_2 = 0$. So there is no solution. (This is a good example of why you can’t stop after one of the coefficient comparisons, even if you think you’ve gotten a solution.)

**Third attempt:** Since there was an incongruity when we compared coefficients of $e^{2t}$, we might try to add something to deal with this incongruity. So try
\[
x = bte^{2t} + de^{2t} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^{2t} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} e^{2t}.
\]

Substituting this into (4), we obtain
\[
\begin{pmatrix} 2b_1 \\ 2b_2 \end{pmatrix} te^{2t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{2t} + \begin{pmatrix} 2d_1 \\ 2d_2 \end{pmatrix} e^{2t} = \begin{pmatrix} b_1 + b_2 \\ 4b_1 - 2b_2 \end{pmatrix} te^{2t} + \begin{pmatrix} d_1 + d_2 \\ 4d_1 - 2d_2 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}.
\]

As in the second attempt, the coefficients of $te^{2t}$ force $b_1 = b_2$. Now we compare coefficients of $e^{2t}$. We get
\[
\begin{align*}
  b_1 + 2d_1 &= d_1 + d_2 + 1 \\
  b_2 + 2d_2 &= 4d_1 - 2d_2.
\end{align*}
\]

So $b_1 = b_2 = -d_1 + d_2 + 1$, from the first equation. Substituting this into the second equation, we have
\[
-d_1 + 3d_2 + 1 = 4d_1 - 2d_2,
\]
so that $5d_1 - 5d_2 = 1$. We have no more equations to use, so we can just pick a solution, e.g., $d_1 = 1/5$ and $d_2 = 0$, which then gives $b_1 = b_2 = 4/5$. (It isn’t absurd that we have more than one choice here, because there’s no canonical ‘particular solution’ to the equation (4) anyway – any solution to (4) could equally well be $x_p^{(1)}$. ) We end up with the solution
\[
x^{(1)}_p = \begin{pmatrix} 4/5 \\ 4/5 \end{pmatrix} te^{2t} + \begin{pmatrix} 1/5 \\ 0 \end{pmatrix} e^{2t}.
\]

Finally, the general solution to (1) is
\[
x = C_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 4/5 \\ 4/5 \end{pmatrix} te^{2t} + \begin{pmatrix} 1/5 \\ 0 \end{pmatrix} e^{2t} + x_p^{(2)} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^t.
\]

Let’s break this process down into steps for solving a general equation of the form
\[
x' = Ax + g(t).
\]

1. Solve the associated homogeneous equation, $x' = Ax$.
2. Break $g$ down into a sum of simpler functions $g^{(1)}, \ldots, g^{(n)}$, each of which is a constant vector times a function of $t$. (This function should be an exponential function, a polynomial, a sine or cosine, or some product of these for the method of undetermined coefficients to work.)
3. For each such function, try to find a particular solution to the problem
\[
x' = Ax + g^{(i)}
\]
by substituting for $x$ an appropriate function of $t$ with undetermined vector coefficients. Use exactly the same principles you learned in the single-variable case to pick the function of $t$. By comparing coefficients of various functions of $t$ on both sides, solve for the undetermined coefficients.
4. If $g^{(i)}$ is a vector times $e^{rt}$ and $r$ is an eigenvalue of $A$, then $x = ae^{rt}$ will probably not work. Instead, try
\[
x = ate^{rt} + be^{rt},
\]
where $a$ and $b$ are undetermined vectors. Likewise, if $g^{(i)}$ is a vector times $e^{\alpha t} \cos(\beta t)$ or $e^{\alpha t} \sin(\beta t)$ and $\alpha \pm i\beta$ are complex conjugate eigenvalues of $A$, then for $x$ you should try
\[
x = at e^{\alpha t} \cos(\beta t) + bte^{\alpha t} \sin(\beta t) + ce^{\alpha t} \cos(\beta t) + de^{\alpha t} \sin(\beta t),
\]
where \( a, \ldots, d \) are undetermined vectors. Finally, if \( g^{(i)} \) is a vector times \( te^{rt} \) and \( r \) is a repeated eigenvalue of \( A \), then you should try
\[
\mathbf{x} = a t^2 e^{rt} + b t e^{rt},
\]
where \( a \) and \( b \) are undetermined vectors.

(5) The general solution to the nonhomogeneous equation is:

\[
(\text{general solution to AHE}) + (\text{particular solution to the equation with forcing function } g^{(1)})
\]
\[
+ \cdots + (\text{particular solution to the equation with forcing function } g^{(n)}).
\]