THE METHOD OF UNDETERMINED COEFFICIENTS FOR OF NONHOMOGENEOUS LINEAR SYSTEMS

Consider the system of differential equations

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{2t} \\ -2e^t \end{pmatrix}. \tag{1}$$

By way of analogy, I'm going to call the function $\mathbf{g}$, or other functions in the same position, a "forcing function", even though this isn't necessarily a spring problem.

Let's solve this using the method of undetermined coefficients. The general solution is of the form

$$\mathbf{x} = \mathbf{x}_C + \mathbf{x}_P$$

where $\mathbf{x}_C$ is the general solution to the associated homogeneous equation, and $\mathbf{x}_P$ is a particular solution to the equation (1).

So we start by solving the associated homogeneous equation

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}. \tag{2}$$

The characteristic polynomial is

$$\det(A - \lambda I) = (1 - \lambda)(-2 - \lambda) - 1 \cdot 4 = \lambda^2 + \lambda - 6,$$

which factors as $(\lambda + 3)(\lambda - 2)$. So the eigenvalues are $\lambda = -3$ and 2.

An eigenvector ξ for $\lambda = -3$ will satisfy

$$(A + 3I)\xi = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \xi = 0.$$

One such vector is $\binom{1}{-4}$.

An eigenvector for $\lambda = 2$ will satisfy

$$(A - 2I)\xi = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} = 0.$$

One such vector is $\binom{1}{1}$.

So the general solution to (2) is

$$\mathbf{x}_C = C_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

We now find a particular solution to (1). First, let's rewrite (1) as

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^t. \tag{3}$$

This can be split into two nonhomogeneous equations with the same AHE:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \tag{4}$$

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^{t} \tag{5}$$

Let $\mathbf{x}_P^{(1)}$ be a particular solution to (4), and let $\mathbf{x}_P^{(2)}$ be a particular solution to (5). Then $\mathbf{x}_P^{(1)} + \mathbf{x}_P^{(2)}$ is a particular solution to (3). (Do you see why? Try substituting this into (3) and using what you know about $\mathbf{x}_P^{(1)}$ and $\mathbf{x}_P^{(2)}$. This is similar to the reason why the sum of two solutions to a linear homogeneous equation is also a solution to the same equation.) So the general solution to (1) can be written in the form

$$\mathbf{x} = \mathbf{x}_C + \mathbf{x}_P^{(1)} + \mathbf{x}_P^{(2)}.$$

Next, we find $\mathbf{x}_P^{(2)}$. The forcing function in (5) is a multiple of e^t (by a vector!), so we should try the function with undetermined (vector-valued!) coefficients,

$$\mathbf{x} = \mathbf{a}e^t = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t.$$

Let's substitute this into (5). We get

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t = \begin{pmatrix} a_1 + a_2 \\ 4a_1 - 2a_2 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^t.$$

(You might be tempted to solve for t here, but that's the wrong move! The two sides are equal as functions of t, so we can just compare the coefficients of e^t and solve for a_1 and a_2.) Comparing the coordinates, we get the system of equations

$$\begin{aligned} a_1 &= a_1 + a_2, \\ a_2 &= 4a_1 - 2a_2 + 2. \end{aligned}$$

(Be sure you understand this step, since it's the sort of vector manipulation you'll need to do quite a bit.) The solution is $a_2 = 0$, $a_1 = 1/2$. So our particular solution to (5) is

$$\mathbf{x}_P^{(2)} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^t. \tag{6}$$

(This illustrates what you should be doing in general. If the forcing function is a vector times e^{rt}, try an unknown vector times e^{rt}. If it's a vector times $\sin(rt)$ or $\cos(rt)$, try an unknown vector times $\sin(rt)$ plus an unknown vector times $\cos(rt)$. If it's a vector times a polynomial of degree d, try a polynomial of degree d with unknown vector coefficients. If it's a product of these, try a product of the appropriate things, just as you would in the single-equation case. As always, don't be afraid to try something and get it wrong – you can usually see what you missed after doing the calculation, and correct accordingly.)

Finally, we have to find $\mathbf{x}_P^{(1)}$. Of course, this is a little subtler since the forcing function in (4) is a multiple of e^{2t}, and 2 is a root of the characteristic polynomial. I'll do multiple attempts to show you what happens.

First attempt: We try

$$\mathbf{x} = \mathbf{b}e^{2t} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{2t}.$$

Plugging this into (4), we get

$$\begin{pmatrix} 2b_1 \\ 2b_2 \end{pmatrix} e^{2t} = \begin{pmatrix} b_1 + b_2 \\ 4b_1 - 2b_2 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}.$$

Comparing the coefficients of e^{2t} on both sides, we get the system of equations

$$\begin{aligned} 2b_1 &= b_1 + b_2 + 1, \\ 2b_2 &= 4b_1 - 2b_2. \end{aligned}$$

It's not too hard to see that this system has no solutions. So this choice of $\mathbf{x}$ didn't work.

Here's one way of thinking about what just happened. For certain values of $\mathbf{b}$ – namely, for $\mathbf{b}$ a scalar multiple of the eigenvector $\binom{1}{1}$ – $\mathbf{b}e^{2t}$ is a solution to the associated homogeneous equation. Substituting any of these functions into $\mathbf{x}' - A\mathbf{x}$ gives zero. So substituting an arbitrary function $\mathbf{b}e^{2t}$ into $\mathbf{x}' - A\mathbf{x}$ gives just a one-dimensional space of outputs, where we sort of need a two-dimensional one. You can check that if $\mathbf{x} = \mathbf{b}e^{2t}$, then $\mathbf{x}' - A\mathbf{x}$ is a scalar multiple of $\binom{1}{1}e^{2t}$. If our forcing function was also a scalar multiple of $\binom{1}{1}e^{2t}$, we'd be able to solve for $\mathbf{x}$, but in general, this is too little space to move in.

Second attempt: Inspired by the single-variable case, let

$$\mathbf{x} = \mathbf{b}te^{2t} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^{2t}.$$

Substituting this into (4), we obtain

$$\begin{pmatrix} 2b_1 \\ 2b_2 \end{pmatrix} te^{2t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{2t} = \begin{pmatrix} b_1 + b_2 \\ 4b_1 - 2b_2 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}.$$

Comparing the coefficients of te^{2t}, we get

$$\begin{aligned} 2b_1 &= b_1 + b_2, \\ 2b_2 &= 4b_1 - 2b_2. \end{aligned}$$

These equations are satisfied whenever $b_1 = b_2$. However, comparing the coefficients of e^{2t}, we also must have $b_1 = 1$ and $b_2 = 0$. So there is no solution. (This is a good example of why you can't stop after one of the coefficient comparisons, even if you think you've gotten a solution.)

Third attempt: Since there was an incongruity when we compared coefficients of e^{2t}, we might try to add something to deal with this incongruity. So try

$$\mathbf{x} = \mathbf{b}te^{2t} + \mathbf{d}e^{2t} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} te^{2t} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} e^{2t}.$$

Substituting this into (4), we obtain

$$\begin{pmatrix} 2b_1 \\ 2b_2 \end{pmatrix} te^{2t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} e^{2t} + \begin{pmatrix} 2d_1 \\ 2d_2 \end{pmatrix} e^{2t} = \begin{pmatrix} b_1 + b_2 \\ 4b_1 - 2b_2 \end{pmatrix} te^{2t} + \begin{pmatrix} d_1 + d_2 \\ 4d_1 - 2d_2 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}.$$

As in the second attempt, the coefficients of te^{2t} force $b_1 = b_2$. Now we compare coefficients of e^{2t}. We get

$$\begin{aligned} b_1 + 2d_1 &= d_1 + d_2 + 1 \\ b_2 + 2d_2 &= 4d_1 - 2d_2. \end{aligned}$$

So $b_1 = b_2 = -d_1 + d_2 + 1$, from the first equation. Substituting this into the second equation, we have

$$-d_1 + 3d_2 + 1 = 4d_1 - 2d_2,$$

so that $5d_1 - 5d_2 = 1$. We have no more equations to use, so we can just pick a solution, e.g., $d_1 = 1/5$ and $d_2 = 0$, which then gives $b_1 = b_2 = 4/5$. (It isn't absurd that we have more than one choice here, because there's no canonical 'particular solution' to the equation (4) anyway – any solution to (4) could equally well be $\mathbf{x}_P^{(1)}$.) We end up with the solution

$$\mathbf{x}_P^{(1)} = \begin{pmatrix} 4/5 \\ 4/5 \end{pmatrix} te^{2t} + \begin{pmatrix} 1/5 \\ 0 \end{pmatrix} e^{2t}. \tag{7}$$

Finally, the general solution to (1) is

$$\mathbf{x} = C_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 4/5 \\ 4/5 \end{pmatrix} te^{2t} + \begin{pmatrix} 1/5 \\ 0 \end{pmatrix} e^{2t} + \mathbf{x}_P^{(2)} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{t}.$$

Let's break this process down into steps for solving a general equation of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t).$$

(1) Solve the associated homogeneous equation, $\mathbf{x}' = A\mathbf{x}$.
(2) Break $\mathbf{g}$ down into a sum of simpler functions $\mathbf{g}^{(1)}, \ldots, \mathbf{g}^{(n)}$, each of which is a constant vector times a function of t. (This function should be an exponential function, a polynomial, a sine or cosine, or some product of these for the method of undetermined coefficients to work.)
(3) For each such function, try to find a particular solution to the problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}^{(i)}$$

by substituting for $\mathbf{x}$ an appropriate function of t with undetermined vector coefficients. Use exactly the same principles you learned in the single-variable case to pick the function of t. By comparing coefficients of various functions of t on both sides, solve for the undetermined coefficients.
(4) If $\mathbf{g}^{(i)}$ is a vector times e^{rt} and r is an eigenvalue of A, then $\mathbf{x} = \mathbf{a}e^{rt}$ will probably not work. Instead, try

$$\mathbf{x} = \mathbf{a}te^{rt} + \mathbf{b}e^{rt},$$

where $\mathbf{a}$ and $\mathbf{b}$ are undetermined vectors. Likewise, if $\mathbf{g}^{(i)}$ is a vector times $e^{\alpha t}\cos(\beta t)$ or $e^{\alpha t}\sin(\beta t)$ and $\alpha \pm i\beta$ are complex conjugate eigenvalues of A, then for $\mathbf{x}$ you should try

$$\mathbf{x} = \mathbf{a}te^{\alpha t}\cos(\beta t) + \mathbf{b}te^{\alpha t}\sin(\beta t) + \mathbf{c}e^{\alpha t}\cos(\beta t) + \mathbf{d}e^{\alpha t}\sin(\beta t),$$

where $\mathbf{a}, \ldots, \mathbf{d}$ are undetermined vectors. Finally, if $\mathbf{g}^{(i)}$ is a vector times te^{rt} and r is a repeated eigenvalue of A, then you should try

$$\mathbf{x} = \mathbf{a}t^2e^{rt} + \mathbf{b}te^{rt},$$

where $\mathbf{a}$ and $\mathbf{b}$ are undetermined vectors.

(5) The general solution to the nonhomogeneous equation is:

$$\begin{aligned}&(\text{general solution to AHE}) + (\text{particular solution to the equation with forcing function } \mathbf{g}^{(1)}) \\ &\qquad + \cdots + (\text{particular solution to the equation with forcing function } \mathbf{g}^{(n)}).\end{aligned}$$