Week 10 solutions

Assignment 22 had no hand-graded component.

ASSIGNMENT 23.

K. *From the theory of elasticity, if the ends of a horizontal beam (of uniform cross-section and constant density) are supported at the same height in vertical walls, then its vertical displacement $y(x)$ satisfies the Boundary Value Problem*

$$y^{(4)} = -P, \quad y(0) = y(L) = 0,\ y'(0) = y'(L) = 0,$$

where $P > 0$ is a constant depending on the beam's density and rigidity and L is the distance between supporting walls.

(a) *Solve the boundary value problem when $L = 4$ and $P = 24$.*
First, we solve the associated homogeneous equation,

$$y^{(4)} = 0.$$

This can be done by just integrating four times: we get

$$y = C_0x^3 + C_1x^2 + C_2x + C_3.$$

The inhomogeneous equation

$$y^{(4)} = -24$$

can be solved by the method of undetermined coefficients. The right-hand side is a polynomial of degree zero, so we would like to try a polynomial of degree zero for y, i. e., $y = A_0$. But this is a solution to the associated homogeneous equation. To avoid the solutions to the associated homogeneous equation, we must multiply by x^4. Thus, we try $y = A_0x^4$. Taking the fourth derivative gives

$$24A_0 = -24,$$

so that $y = -x^4$ is a particular solution to the inhomogeneous equation.
Thus, the general solution to the inhomogeneous equation is

$$y = -x^4 + C_0x^3 + C_1x^2 + C_2x + C_3.$$

Note that we're not given the values of $y, \dots, y^{(3)}$ at a single value of x, as has usually been the setup in this class. However, we are given four independent pieces of information about the function y, which should be enough to determine the constants. We have

$$y' = -4x^3 + 3C_0x^2 + 2C_1x + C_2.$$

Since $y(0) = y'(0) = 0$, we get $C_3 = C_2 = 0$. Then

$$0 = y(4) = -4^4 + C_0 \cdot 4^3 + C_1 \cdot 4^2 = 4^2(-16 + 4C_0 + C_1),$$
$$‘0 = y'(4) = -4^4 + 3C_0 \cdot 4^2 + 2C_1 \cdot 4 = 8(-32 + 6C_0 + C_1).$$

So we have a system of equations

$$C_1 + 4C_0 = 16,$$
$$C_1 + 6C_0 = 32.$$

The solution is $C_0 = 8$, $C_1 = -16$. Thus, the equation for y is

$$y = -x^4 + 8x^3 - 16x^2 = -x^2(x-4)^2.$$

(b) *Show that the maximum displacement occurs at the center of the beam $x = L/2 = 2$.*

The derivative of y is

$$y' = -4x^3 + 24x^2 - 32x = -4x(x^2 - 6x + 8) = -4x(x-2)(x-4).$$

So there are three critical points for y, at $x = 0$, $x = 2$, and $x = 4$. By looking at the graph of y, or by checking the sign of $y'' = -12x^2 + 48x - 32$ at $x = 2$, we can see that $x = 2$ is a minimum of y, and thus is the maximum displacement for $0 \le x \le L$. (The other two critical points are local maxima of y, and correspond to where the beam is attached to the walls, where it isn't displaced at all!)

ASSIGNMENT 24.

6.1.5(b) *Find the Laplace transform of $f(t) = t^2$.*

This is a repeated integration by parts.

$$\begin{aligned}
\mathcal{L}[t^2] &= \int_0^\infty t^2 e^{-st}\,dt \quad \text{(integrate by parts with } u = t^2,\ v' = e^{-st}) \\
&= \left[-t^2\frac{e^{-st}}{s}\right]_0^\infty + \int_0^\infty 2t\frac{e^{-st}}{s}\,dt \quad \text{(integrate by parts with } u = 2t,\ v' = e^{-st}/s) \\
&= \left[-t^2\frac{e^{-st}}{s}\right]_0^\infty - \left[2t\frac{e^{-st}}{s^2}\right]_0^\infty + \int_0^\infty 2\frac{e^{-st}}{s^2}\,dt \\
&= \left[-t^2\frac{e^{-st}}{s}\right]_0^\infty - \left[2t\frac{e^{-st}}{s^2}\right]_0^\infty - \left[2\frac{e^{-st}}{s^3}\right]_0^\infty \quad (e^{st} \text{ grows faster than any polynomial in } t \text{ for } s > 0) \\
&= 0 + 0^2 \cdot e^{-s\cdot 0}/s - 0 + 0 \cdot 2e^{-s\cdot 0}/s^2 - 0 + 2e^{-s\cdot 0}/s^3 = \frac{2}{s^3} \quad \text{(for } s > 0).
\end{aligned}$$

8. *Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. Find the Laplace transform of $f(t) = \sinh bt$.*

We use the fact that $\mathcal{L}[e^{bt}] = 1/(s-b)$, which we proved in class. We also use the fact that the Laplace transform is a linear operator. So

$$\begin{aligned}\mathcal{L}[\sinh bt] &= \mathcal{L}\left[\frac{e^{bt}-e^{-bt}}{2}\right] \\ &= \frac{\mathcal{L}[e^{bt}]-\mathcal{L}[e^{-bt}]}{2} \\ &= \frac{\frac{1}{s-b}-\frac{1}{s+b}}{2} \\ &= \frac{(s+b)-(s-b)}{2(s-b)(s+b)} \\ &= \frac{b}{s^2-b^2}.\end{aligned}$$

You might want to compare this with the formula $\mathcal{L}[\sin bt] = \frac{b}{s^2+b^2}$.

15. *Use integration by parts to find the Laplace transform of $f(t) = te^{at}$, where a is a real constant.*

$$\begin{aligned}\mathcal{L}[te^{at}] &= \int_0^\infty te^{at}e^{-st}\,dt \\ &= \int_0^\infty te^{(a-s)t}\,dt \quad \text{(integrate by parts with } u=t,\ v'=e^{(a-s)t}\text{)} \\ &= \left[t\frac{e^{(a-s)t}}{a-s}\right]_0^\infty - \int_0^\infty \frac{e^{(a-s)t}}{a-s}\,dt \\ &= \left[t\frac{e^{(a-s)t}}{a-s}\right]_0^\infty - \left[\frac{e^{(a-s)t}}{(a-s)^2}\right]_0^\infty \\ &= 0 - 0\cdot e^{(a-s)\cdot 0}/(a-s) - 0 + e^{(a-s)\cdot 0}(a-s)^2 = \frac{1}{(s-a)^2} \quad (\text{for } s>a).\end{aligned}$$

ASSIGNMENT 25.

6.2.10. *Find the inverse Laplace transform of*

$$F(s) = \frac{2s-3}{s^2+2s+10}.$$

The denominator has non-real roots, so rather than factoring it into linear terms and using partial fractions, we should try to find inverse Laplace transforms using the formulas

$$\mathcal{L}[e^{at}\sin(bt)] = \frac{b}{(s-a)^2+b^2}, \quad \mathcal{L}[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2+b^2}.$$

We start by completing the square on the bottom:

$$s^2+2s+10 = s^2+2s+1+9 = (s+1)^2+3^2.$$

So we should be able to use the above formulas with $a = -1$ and $b = 3$. We only need to get the numerator, $2s - 3$, in the right form – it should be a linear combination of $(s + 1)$ and 3. We have

$$\frac{2s-3}{s^2+2s+10} = \frac{2s-3}{(s+1)^2+3^2} = \frac{2(s+1)-5}{(s+1)^2+3^2} = 2\frac{s+1}{(s+1)^2+3^2} - \frac{5}{3}\frac{3}{(s+1)^2+3^2}.$$

By the linearity of the Laplace transform, the inverse Laplace transform is

$$f(t) = 2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t).$$

6.2.21. *Use the Laplace transform to solve the initial value problem*

$$y'' - 2y' + 2y = \cos(t), \quad y(0) = 1, \; y'(0) = 0.$$

Let $Y = \mathcal{L}[y]$. Then

$$\mathcal{L}[y'] = sY - y(0) = sY - 1,$$
$$\mathcal{L}[y''] = s^2Y - sy(0) - y'(0) = s^2Y - s.$$

We also know that the Laplace transform of $\cos(t)$ is $\frac{s}{s^2+1}$. Taking the Laplace transform of the whole equation, we have

$$s^2Y - s - 2sY + 2 + 2Y = \frac{s}{s^2+1}.$$

We now start rearranging:

$$\begin{aligned} s^2Y - 2sY + 2Y &= \frac{s}{s^2+1} + s - 2 \\ Y &= \frac{s}{(s^2+1)(s^2-2s+2)} + \frac{s-2}{s^2-2s+2}. \end{aligned}$$

It is probably possible to find an inverse Laplace transform for the second fraction, using the method of the previous problem. However, we need to split the first fraction up using partial fractions. If

$$\frac{s}{(s^2+1)(s^2-2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2-2s+2},$$

then

$$s = (As+B)(s^2-2s+2)+(Cs+D)(s^2+1) = (A+C)s^3+(-2A+B+D)s^2+(2A-2B+C)s+(2B+D).$$

This gives a system of equations that we can solve for A, B, C, and D. We obtain

$$\frac{s}{(s^2+1)(s^2-2s+2)} = \frac{1}{5}\frac{s-2}{s^2+1} + \frac{1}{5}\frac{-s+4}{s^2-2s+2}.$$

Therefore,

$$\begin{aligned} Y &= \frac{s}{(s^2+1)(s^2-2s+2)} + \frac{s-2}{s^2-2s+2} = \frac{1}{5}\frac{s-2}{s^2+1} + \frac{1}{5}\frac{4s-6}{s^2-2s+2} \\ &= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} + \frac{4}{5}\frac{s-1}{(s-1)^2+1} - \frac{2}{5}\frac{1}{(s-1)^2+1}. \end{aligned}$$

Taking the inverse Laplace transform gives

$$y(t) = \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) + \frac{4}{5}e^t\cos(t) - \frac{2}{5}e^t\sin(t).$$