Week 11 solutions

ASSIGNMENT 26.

6.3.1. *Sketch the graph of the given function on the interval* $t \geq 0$.

$$g(t) = u_1(t) + 2u_3(t) - 6u_4(t).$$

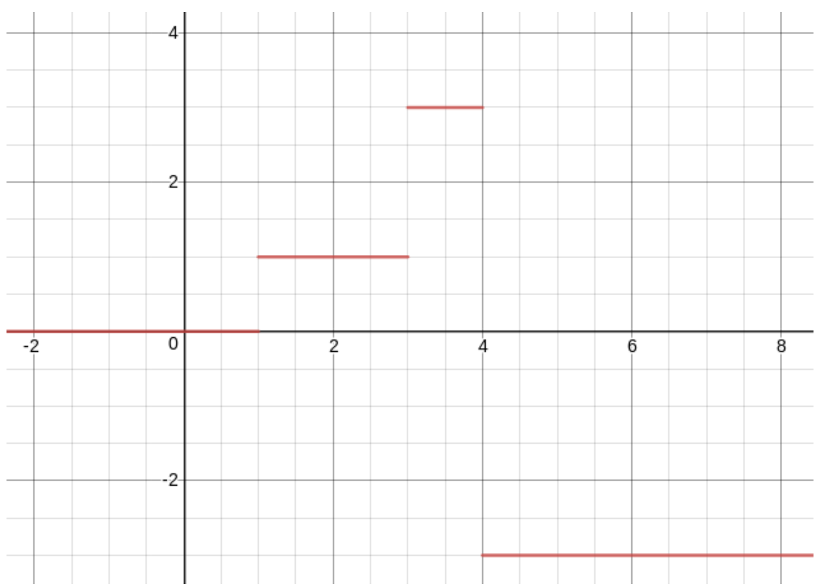
We can write g as a piecewise function:

$$g(t) = \begin{cases} 0 & 0 \leq t < 1, \\ 1 & 1 \leq t < 3, \\ 3 & 3 \leq t < 4, \\ -3 & 4 \leq t. \end{cases}$$

I entered this into Desmos as

$$y = \{x < 1 : 0,\ 1 \leq x < 3 : 1,\ 3 \leq x < 4 : 3,\ 4 \leq x : -3\}$$

and got the graph:



2. *Sketch the graph of the given function on the interval* $t \geq 0$.

$$g(t) = (t - 3)u_2(t) - (t - 2)u_3(t).$$

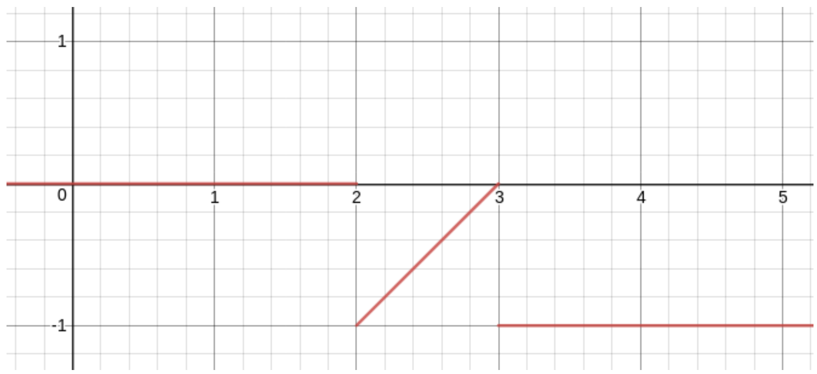
As a piecewise function,

$$g(t) = \begin{cases} 0 & 0 \leq t < 2, \\ t-3 & 2 \leq t < 3, \\ -1 & 3 \leq t. \end{cases}$$

(The last piece, for $t \geq 3$, is $(t-3)-(t-2) = -1$.) The graph of

$$y = \{x < 2 : 0,\ 2 \leq x < 3 :\ x-3,\ 3 \leq x :\ -1\}$$

is:



18. *Find the Laplace transform of the given function.*

$$f(t) = t - u_1(t)(t-1), \quad t \geq 0.$$

Recall the translation property,

$$\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

So for this f,

$$\mathcal{L}[f] = \mathcal{L}[t] - \mathcal{L}[u_1(t)(t-1)] = \mathcal{L}[t] - e^{-s}\mathcal{L}[t] = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

You could also get the same answer if you started by rewriting f as a sum of shifts:

$$f(t) = t - \mathrm{sh}_1(t).$$

21. *Find the inverse Laplace transform of the given function.*

$$F(s) = \frac{2(s-1)e^{-2s}}{s^2-2s+2}.$$

Let

$$G(s) = \frac{2(s-1)}{s^2-2s+2}.$$

We will start by finding $\mathcal{L}^{-1}[G(s)]$. We have

$$G(s) = 2\frac{s-1}{(s-1)^2+1},$$

so its inverse Laplace transform is

$$g(t) = 2e^t \cos t.$$

Now, since $F(s) = e^{-2s}G(s)$,

$$\mathcal{L}^{-1}[F(s)] = \mathrm{sh}_2(g(t)) = \mathrm{sh}_2(2e^t \cos(t)).$$

This could also be written as

$$\mathcal{L}^{-1}[F] = u_2(t) \cdot 2e^{t-2} \cos(t-2),$$

or as

$$\mathcal{L}^{-1}[F] = \begin{cases} 0 & t < 2 \\ 2e^{t-2}\cos(t-2) & 2 \leq t. \end{cases}$$

23. *Find the inverse Laplace transform of the given function.*

$$F(s) = \frac{(s-2)e^{-s}}{s^2 - 4s + 3}.$$

Again, let

$$G(s) = \frac{s-2}{s^2 - 4s + 3}.$$

We begin by finding the inverse Laplace transform of G. Unlike in the previous problem, the denominator of G factors over the reals as $(s-1)(s-3)$. So we should find a partial fraction decomposition

$$\frac{s-2}{s^2-4s+3} = \frac{A}{s-1} + \frac{B}{s-3}.$$

Multiplying through:

$$s - 2 = A(s-3) + B(s-1).$$

If $s = 3$, we get $1 = 2B$, so $B = 1/2$. If $s = 1$, we get $-1 = -2A$, so $A = 1/2$. Thus,

$$G(s) = \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s-3}.$$

So

$$g(t) = \mathcal{L}^{-1}[G] = \frac{1}{2}e^t + \frac{1}{2}e^{3t}.$$

Since $F(s) = e^{-s}G(s)$,

$$\mathcal{L}^{-1}[F] = \mathrm{sh}_1 \mathcal{L}^{-1}[G] = \mathrm{sh}_1\left(\frac{1}{2}e^t + \frac{1}{2}e^{3t}\right).$$

This could also be written as

$$\mathcal{L}^{-1}[F] = u_1(t) \cdot \left(\frac{1}{2}e^{t-1} + \frac{1}{2}e^{3(t-1)}\right),$$

or as

$$\mathcal{L}^{-1}[F] = \begin{cases} 0 & 0 \leq t < 1 \\ \frac{1}{2}e^{t-1} + \frac{1}{2}e^{3(t-1)} & 1 \leq t. \end{cases}$$

Assignment 27 had no hand-graded component.