Week 12 solutions

ASSIGNMENT 28.

6.5.14(a,b). *Consider the initial value problem*

$$y'' + \gamma y' + y = \delta(t-1), \quad y(0), y'(0) = 0,$$

where γ is the damping coefficient (or resistance).

(a) *Let $\gamma = 1$. Find the solution of the initial value problem and plot its graph.* If $Y = \mathcal{L}[y]$, then the Laplace transform of the equation is

$$s^2 Y + sY + Y = e^{-s},$$

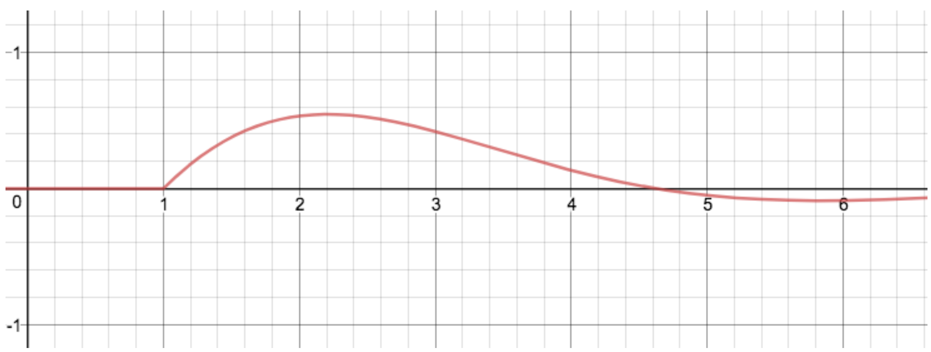
or

$$Y = \frac{e^{-s}}{s^2+s+1} = e^{-s}\left(\frac{1}{(s+1/2)^2+3/4}\right).$$

The inverse Laplace transform is

$$y = \mathrm{sh}_1\left(\frac{2}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}t/2)\right) = \begin{cases} 0 & 0 \le t < 1 \\ \frac{2}{\sqrt{3}}e^{-(t-1)/2}\sin(\sqrt{3}(t-1)/2) & 1 \le t. \end{cases}$$

Here is a graph:



(b) *Find the time t_1 at which the solution attains its maximum value. Also find the maximum value y_1 of the solution.* Using Desmos, I found that the maximum is at $(2.209, 0.546)$.

L. *Use Laplace transforms to find a particular solution, $y_p(t)$, of*

$$y'' + 4y = 20e^t.$$

Since any particular solution will do, we can pick the initial conditions freely. For example, let $y(0) = y'(0) = 0$. Then if $Y = \mathcal{L}[y]$, the Laplace transform of the equation is

$$s^2 Y + 4Y = \frac{20}{s-1},$$

which simplifies to

$$Y = \frac{20}{(s-1)(s^2+4)}.$$

Using partial fractions, we have

$$Y = \frac{4}{s-1} + \frac{-4s-4}{s^2+4}.$$

The inverse Laplace transform is

$$y = 4e^t - 4\cos(2t) - 2\sin(2t).$$

ASSIGNMENT 29.

6.6.4. *Find the Laplace transform of the function*

$$f(t) = \int_0^t (t-\tau)^2 \cos(2\tau)\, d\tau.$$

This function is a convolution:

$$f(t) = t * \cos(2t).$$

So

$$\mathcal{L}[f] = \mathcal{L}[t] \cdot \mathcal{L}[\cos(2t)] = \frac{1}{s} \cdot \frac{s}{s^2+4} = \frac{s}{s(s^2+4)}.$$

6.6.5. *Find the Laplace transform of the function*

$$f(t) = \int_0^t e^{-(t-\tau)} \sin(\tau)\, d\tau.$$

Again, write

$$f(t) = e^{-t} * \sin(t).$$

Then

$$\mathcal{L}[f] = \mathcal{L}[e^{-t}] \cdot \mathcal{L}[\sin(t)] = \frac{1}{s+1} \cdot \frac{1}{s^2+1} = \frac{1}{(s+1)(s^2+1)}.$$

ASSIGNMENT 30.

7.2.22. *Verify that the given vector satisfies the given differential equation.*

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}.$$

We calculate that

$$\mathbf{x}' = 2 \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$$

and that

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 3 \cdot 4 - 2 \cdot 2 \\ 2 \cdot 4 - 2 \cdot 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}.$$

So the given $\mathbf{x}$ satisfies the differential equation.

7.2.23. *Verify that the given vector satisfies the given differential equation.*

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^t.$$

We have

$$\mathbf{x}' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^t = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t,$$

and

$$\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t = \begin{pmatrix} 2 \cdot 1 - 1 \cdot 0 \\ 3 \cdot 1 - 2 \cdot 0 \end{pmatrix} e^t + \begin{pmatrix} 2 \cdot 2 - 1 \cdot 2 \\ 3 \cdot 2 - 2 \cdot 2 \end{pmatrix} te^t = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^t.$$

M. *Tank 1 initially contains 50 gals of water with 10 oz of salt in it, while Tank 2 initially contains 20 gals of water with 15 oz of salt in it. Water containing 2 oz/gal of salt flows into Tank 1 at a rate of 5 gal/min and the well-stirred mixture flows from Tank 1 into Tank 2 at the same rate of 5 gal/min. The solution in Tank 2 flows out to the ground at a rate of 5 gal/min. If $x_1(t)$ and $x_2(t)$ represent the number of ounces of salt in Tank 1 and Tank 2, respectively, set up but do not solve an initial value problem describing this system.*

The intake into Tank 1 adds 10 oz of salt per minute to it. 5 gallons of water from Tank 1, which contain $5x_1/50 = x_1/10$ ounces of salt, flow from Tank 1 to Tank 2 every minute. So the differential equation for x_1 is

$$x_1' = 10 - x_1/10.$$

Tank 2 receives the aforementioned $x_1/10$ ounces of salt, and loses $5x_2/20 = x_2/4$ ounces of salt from the outflow, so its differential equation is

$$x_2' = x_1/10 - x_2/4.$$

(Notice that the volume of water in each tank is constant – if it weren't, the denominators would also have to change over time The initial values are

$$x_1(0) = 10, x_2(0) = 15.$$

This system is an acceptable solution, but we could also write the system as a vector-valued differential equation. If $\mathbf{x} = \binom{x_1}{x_2}$, then the equation is

$$\mathbf{x}' = \begin{pmatrix} -1/10 & 0 \\ 1/10 & -1/4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10 \\ 0 \end{pmatrix}.$$