Week 13 solutions

ASSIGNMENT 31.

7.3.17. *Find all eigenvalues and eigenvectors of the matrix* $\left(\begin{smallmatrix} 3 & -2 \\ 4 & -1 \end{smallmatrix}\right)$.

The characteristic polynomial is

$$\begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 8 = \lambda^2 - 2\lambda + 5.$$

The eigenvalues are the roots of this polynomial, i. e., $\lambda = 1 \pm 2i$. First take $\lambda = 1+2i$. Eigenvectors are solutions to

$$\begin{pmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

This system of equations are redundant, so it has infinitely many solutions. More precisely, if ξ is any eigenvector, any scalar multiple of it will also be an eigenvector. To find an eigenvector, we solve the equation coming from the top row:

$$(2-2i)\xi_1 - 2\xi_2 = 0,$$

which means $\xi_1 = \xi_2/(1-i) = \frac{1+i}{2}\xi_2$. So the set of eigenvectors corresponding to this eigenvalue is the set of (nonzero) scalar multiples (by complex numbers) of

$$\xi = \begin{pmatrix} \frac{1+i}{2} \\ 1 \end{pmatrix}.$$

The second set of eigenvectors can be found by repeating this process for the eigenvalue $1-2i$. Alternatively, since the matrix has real entries and complex conjugate eigenvalues, the eigenvectors for $1-2i$ are precisely the complex conjugates of the eigenvectors for $1+2i$. So they are the nonzero scalar multiples of

$$\xi = \begin{pmatrix} \frac{1-i}{2} \\ 1 \end{pmatrix}.$$

7.3.20. *Find all eigenvalues and eigenvectors of the matrix* $\left(\begin{smallmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{smallmatrix}\right)$.

The characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{vmatrix} = \lambda^2 - 4.$$

So the eigenvalues are ± 2. When $\lambda = 2$, we need to solve

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \xi = 0.$$

The solutions are the scalar multiples of $\xi = \binom{\sqrt{3}}{1}$. When $\lambda = -2$, we need to solve

$$\begin{pmatrix} -3 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \xi = 0.$$

The solutions are the scalar multiples of $\xi = \binom{1}{\sqrt{3}}$.

ASSIGNMENT 32.

7.5.1. *Find the general solution of the system of equations and describe the behavior of the solution as $t \to \infty$. Draw a direction field and plot a few trajectories of the system.*

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}.$$

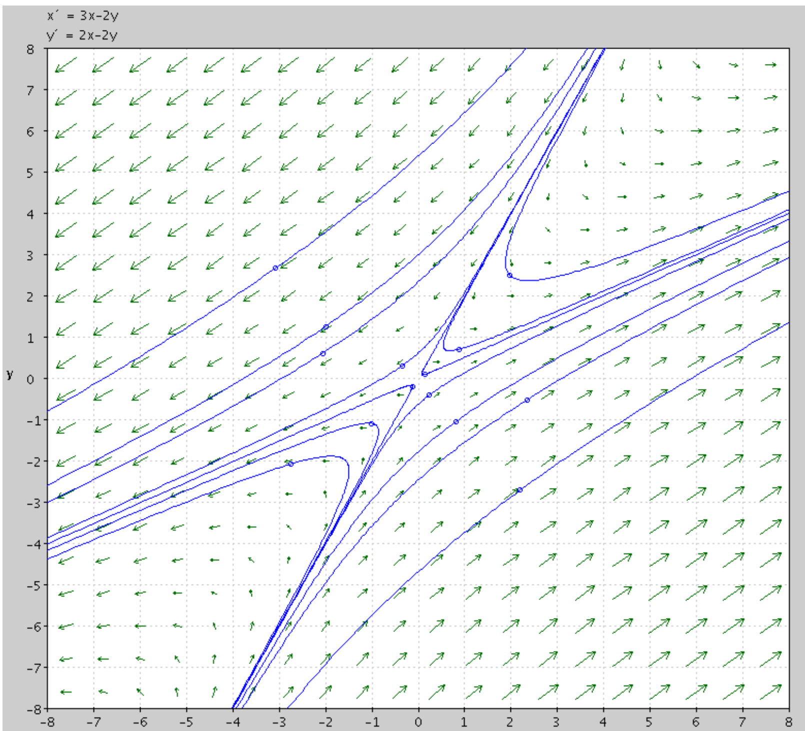
The eigenvalues are the solutions to

$$0 = \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2,$$

which are $\lambda = 2$, $\lambda = -1$. Corresponding to $\lambda = 2$, we have the eigenvector $\binom{2}{1}$. Corresponding to $\lambda = -1$, we have the eigenvector $\binom{1}{2}$. So the general solution is

$$\mathbf{x} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}.$$

As $t \to \infty$, the e^{-t} term vanishes and the e^{2t} term gets large. The solution approaches infinity, along a line parallel to the vector $\binom{2}{1}$. Here is a graph from `pplane`:



7.5.4. *Find the general solution of the system of equations and describe the behavior of the solution as $t \to \infty$. Draw a direction field and plot a few trajectories of the system.*

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}.$$

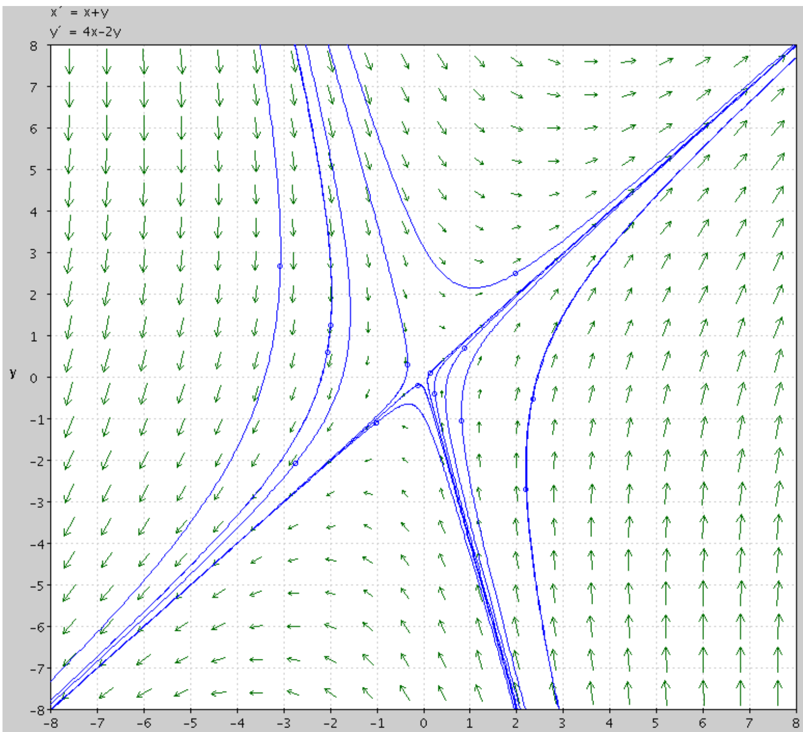
The eigenvalues are the solutions to

$$0 = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 + \lambda - 6,$$

which are $\lambda = 2$, $\lambda = -3$. Corresponding to $\lambda = 2$, we have the eigenvector $\binom{1}{1}$. Corresponding to $\lambda = -3$, we have the eigenvector $\binom{1}{-4}$. So the general solution is

$$\mathbf{x} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.$$

As in the previous problem, the e^{2t} term dominates for large t, and the solutions go to infinity parallel to $\binom{1}{1}$. Here is a graph:



N. *If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent solutions to the 2×2 system $\mathbf{x}' = A\mathbf{x}$, then the matrix $\mathbf{\Phi}(t) = (\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t))$ is called a* Fundamental Matrix for the system. Find a Fundamer

$$\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}.$$

The eigenvalues are the solutions to

$$0 = \begin{vmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{vmatrix} = \lambda^2 + 2\lambda,$$

which are $\lambda = 0$, $\lambda = -2$. Corresponding to $\lambda = 0$ is the eigenvector $\binom{3}{4}$. Corresponding to $\lambda = -2$ is the eigenvector $\binom{1}{2}$. These give two linearly independent solutions:

$$\mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} (= \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{0t}), \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}.$$

A fundamental matrix is given by

$$\mathbf{Phi}(t) = \begin{pmatrix} 3 & e^{-2t} \\ 4 & 2e^{-2t} \end{pmatrix}.$$

Of course, this is not the only fundamental matrix. If we chose a different fundamental set of solutions, we'd get a different matrix.

ASSIGNMENT 33.

7.6.2. *Express the solution of the given system of equations in terms of real-valued functions. Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $t \to \infty$.*

$$\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}.$$

The characteristic polynomial is

$$\begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5,$$

whose roots are $-1 \pm 2i$. An eigenvector for the eigenvalue $-1+2i$ is $\binom{2i}{1}$. This gives us the complex-valued solution

$$\mathbf{x} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{(-1+2i)t} = \begin{pmatrix} 2ie^{-t}(\cos(2t) + i\sin(2t)) \\ e^{-t}\cos(2t) + i\sin(2t) \end{pmatrix} = \begin{pmatrix} -2e^{-t}\sin(2t) + i \cdot 2e^{-t}\cos(2t) \\ e^{-t}\cos(2t) + ie^{-t}\sin(2t) \end{pmatrix}.$$

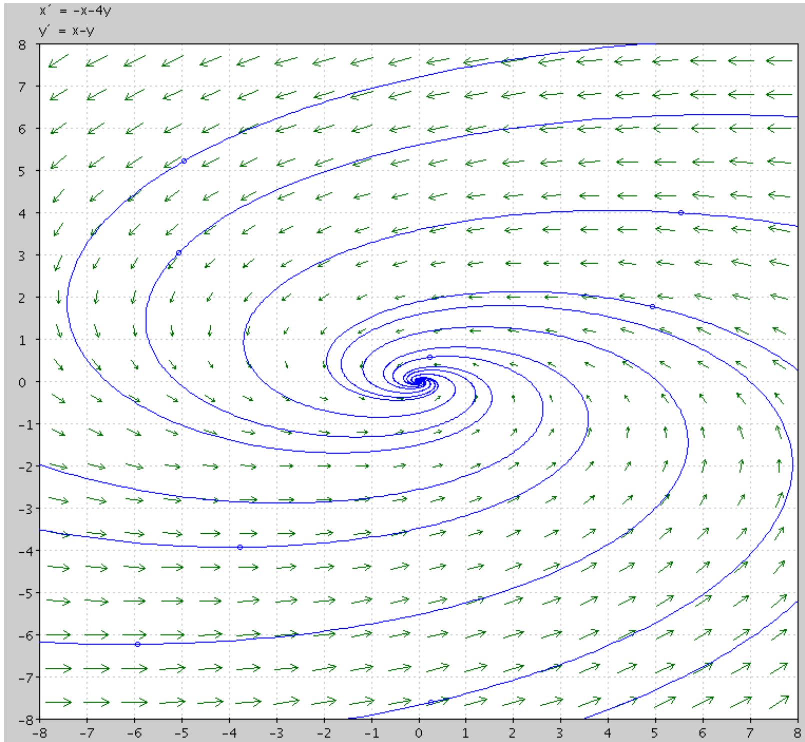
As the differential equation is linear, homogeneous, and has real coefficients, the real and imaginary part of this are also solutions:

$$\mathbf{x}^{(1)} = \begin{pmatrix} -2e^{-t}\sin(2t) \\ e^{-t}\cos(2t) \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{pmatrix}.$$

The general real-valued solution is

$$\mathbf{x} = C_1 \begin{pmatrix} -2e^{-t}\sin(2t) \\ e^{-t}\cos(2t) \end{pmatrix} + C_2 \begin{pmatrix} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{pmatrix}.$$

Here is a graph:



As you can see, the solutions spiral towards zero as $t \to \infty$. So zero is what's called an asymptotically stable spiral point.

Notice that we did not need to use both eigenvalues to find the complete set of real-valued solutions. In fact, the complex-valued solution given by an eigenvector for the second eigenvalue would just be the complex conjugate of the first complex-valued solution we found (or a scalar multiple thereof). So its real and imaginary part would give us no new information.

7.6.6. *Express the solution of the given system of equations in terms of real-valued functions. Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $t \to \infty$.*

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}.$$

The characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 2 \\ -5 & -1-\lambda \end{vmatrix} = \lambda^2 + 9,$$

whose roots are $\pm 3i$. An eigenvector for the eigenvalue $3i$ is $\binom{1+3i}{5}$. This gives us the complex-valued solution

$$\mathbf{x} = \begin{pmatrix} 1+3i \\ 5 \end{pmatrix} e^{3it} = \begin{pmatrix} (1+3i)(\cos(3t) + i\sin(3t)) \\ 5(\cos(3t) + i\sin(3t)) \end{pmatrix} = \begin{pmatrix} (\cos(3t) - 3\sin(3t)) + i(3\cos(3t) + \sin(3t)) \\ 5\cos(3t) + 5i\sin(3t) \end{pmatrix}.$$

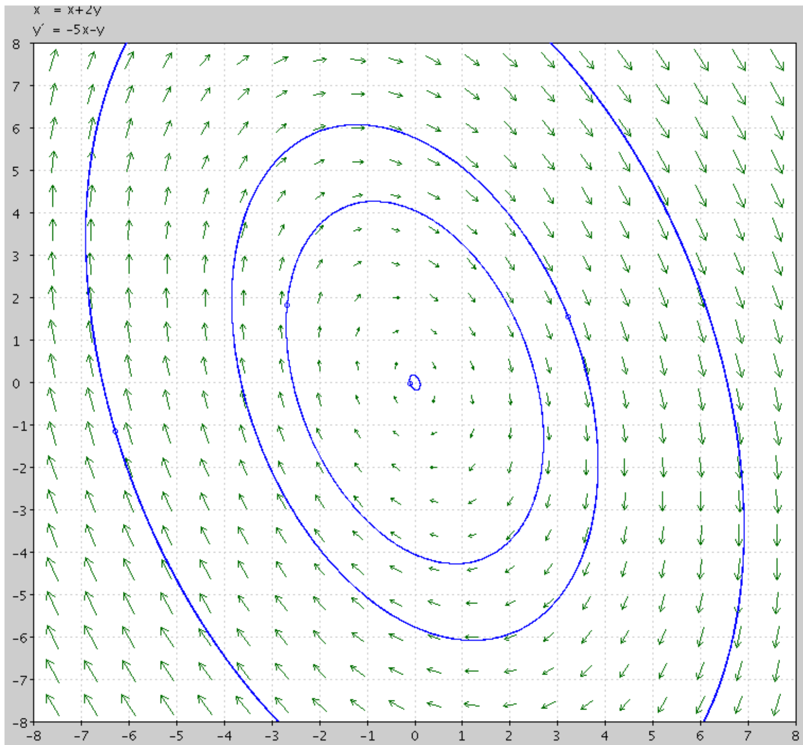
Taking the real and imaginary part of this, we get real-valued solutions

$$\mathbf{x}^{(1)} = \begin{pmatrix} \cos(3t) - 3\sin(3t) \\ 5\cos(3t) \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 3\cos(3t) + \sin(3t) \\ 5\sin(3t) \end{pmatrix}.$$

The general real-valued solution is

$$\mathbf{x} = C_1 \begin{pmatrix} \cos(3t) - 3\sin(3t) \\ 5\cos(3t) \end{pmatrix} + C_2 \begin{pmatrix} 3\cos(3t) + \sin(3t) \\ 5\sin(3t) \end{pmatrix}.$$

Here is a graph:



Solutions orbit around ellipses with $(0, 0)$ as the center.