7.3.17. Find all eigenvalues and eigenvectors of the matrix \((\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix})\).

The characteristic polynomial is
\[
\begin{vmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{vmatrix} = (3 - \lambda)(-1 - \lambda) + 8 = \lambda^2 - 2\lambda + 5.
\]

The eigenvalues are the roots of this polynomial, i.e., \(\lambda = 1 \pm 2i\). First take \(\lambda = 1 + 2i\). Eigenvectors are solutions to
\[
\begin{pmatrix} 3 - (1 + 2i) & -2 \\ 4 & -1 -(1 + 2i) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.
\]

This system of equations are redundant, so it has infinitely many solutions. More precisely, if \(\xi\) is any eigenvector, any scalar multiple of it will also be an eigenvector. To find an eigenvector, we solve the equation coming from the top row:
\[
(2 - 2i)\xi_1 - 2\xi_2 = 0,
\]
which means \(\xi_1 = \xi_2/(1 - i) = \frac{1+i}{2}\xi_2\). So the set of eigenvectors corresponding to this eigenvalue is the set of (nonzero) scalar multiples (by complex numbers) of
\[
\xi = \begin{pmatrix} \frac{1+i}{2} \\ 1 \end{pmatrix}.
\]

The second set of eigenvectors can be found by repeating this process for the eigenvalue \(1 - 2i\). Alternatively, since the matrix has real entries and complex conjugate eigenvalues, the eigenvectors for \(1 - 2i\) are precisely the complex conjugates of the eigenvectors for \(1 + 2i\). So they are the nonzero scalar multiples of
\[
\xi = \begin{pmatrix} \frac{1-i}{2} \\ 1 \end{pmatrix}.
\]

7.3.20. Find all eigenvalues and eigenvectors of the matrix \((\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix})\).

The characteristic polynomial is
\[
\begin{vmatrix} 1 - \lambda & \sqrt{3} \\ \sqrt{3} & -1 - \lambda \end{vmatrix} = \lambda^2 - 4.
\]

So the eigenvalues are \(\pm 2\). When \(\lambda = 2\), we need to solve
\[
\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix}\xi = 0.
\]
The solutions are the scalar multiples of $\xi = (\sqrt{3})$. When $\lambda = -2$, we need to solve

$$\begin{pmatrix} -3 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \xi = 0.$$ 

The solutions are the scalar multiples of $\xi = (\frac{1}{\sqrt{3}})$.

ASSIGNMENT 32.

7.5.1. Find the general solution of the system of equations and describe the behavior of the solution as $t \to \infty$. Draw a direction field and plot a few trajectories of the system.

$$x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x.$$ 

The eigenvalues are the solutions to

$$0 = \begin{vmatrix} 3 - \lambda & -2 \\ 2 & -2 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 2,$$

which are $\lambda = 2$, $\lambda = -1$. Corresponding to $\lambda = 2$, we have the eigenvector $(\frac{2}{1})$. Corresponding to $\lambda = -1$, we have the eigenvector $(\frac{1}{2})$. So the general solution is

$$x = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}.$$ 

As $t \to \infty$, the $e^{-t}$ term vanishes and the $e^{2t}$ term gets large. The solution approaches infinity, along a line parallel to the vector $(\frac{2}{1})$. Here is a graph from pplane:
7.5.4. **Find the general solution of the system of equations and describe the behavior of the solution as \( t \to \infty \). Draw a direction field and plot a few trajectories of the system.**

\[
x' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x}.
\]

The eigenvalues are the solutions to

\[
0 = \begin{vmatrix} 1 - \lambda & 1 \\ 4 & -2 - \lambda \end{vmatrix} = \lambda^2 + \lambda - 6,
\]

which are \( \lambda = 2, \lambda = -3 \). Corresponding to \( \lambda = 2 \), we have the eigenvector \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Corresponding to \( \lambda = -3 \), we have the eigenvector \( \begin{pmatrix} 1 \\ -4 \end{pmatrix} \). So the general solution is

\[
\mathbf{x} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}.
\]

As in the previous problem, the \( e^{2t} \) term dominates for large \( t \), and the solutions go to infinity parallel to \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Here is a graph:

---

N. **If** \( x^{(1)}(t) \) **and** \( x^{(2)}(t) \) **are linearly independent solutions to the** \( 2 \times 2 \) **system** \( x' = Ax \), **then the matrix** \( \Phi(t) = (x^{(1)}(t), x^{(2)}(t)) \) **is called a Fundamental Matrix for the system. Find a Fundamental Matrix** \( \Phi(t) \) **for the system**

\[
x' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{x}.
\]

The eigenvalues are the solutions to

\[
0 = \begin{vmatrix} 4 - \lambda & -3 \\ 8 & -6 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda,
\]
which are \( \lambda = 0, \lambda = -2 \). Corresponding to \( \lambda = 0 \) is the eigenvector \((\frac{3}{4})\). Corresponding to \( \lambda = -2 \) is the eigenvector \((\frac{1}{2})\). These give two linearly independent solutions:

\[
x^{(1)} = \left(\begin{array}{c} 3 \\ 4 \end{array}\right) \left(\begin{array}{c} e^{0t} \\ e^{-2t} \end{array}\right),
\]

\[
x^{(2)} = \left(\begin{array}{c} 1 \\ 2 \end{array}\right) e^{-2t}.
\]

A fundamental matrix is given by

\[
\Phi(t) = \left(\begin{array}{cc} 3 & e^{-2t} \\ 4 & 2e^{-2t} \end{array}\right).
\]

Of course, this is not the only fundamental matrix. If we chose a different fundamental set of solutions, we’d get a different matrix.

**ASSIGNMENT 33.**

7.6.2. Express the solution of the given system of equations in terms of real-valued functions. Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as \( t \to \infty \).

\[
x' = \left(\begin{array}{cc} -1 & -4 \\ 1 & -1 \end{array}\right)x.
\]

The characteristic polynomial is

\[
\begin{vmatrix} -1 - \lambda & -4 \\ 1 & -1 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5,
\]

whose roots are \(-1 \pm 2i\). An eigenvector for the eigenvalue \(-1 + 2i\) is \((\frac{2i}{1})\). This gives us the complex-valued solution

\[
x = \left(\begin{array}{c} 2i \\ 1 \end{array}\right) e^{(-1+2i)t} = \left(\begin{array}{c} 2te^{-t}(\cos(2t) + i\sin(2t)) \\ e^{-t}\cos(2t) + i\sin(2t) \end{array}\right) = \left(\begin{array}{c} -2e^{-t}\sin(2t) + i \cdot 2e^{-t}\cos(2t) \\ e^{-t}\cos(2t) + ie^{-t}\sin(2t) \end{array}\right).
\]

As the differential equation is linear, homogeneous, and has real coefficients, the real and imaginary part of this are also solutions:

\[
x^{(1)} = \left(\begin{array}{c} -2e^{-t}\sin(2t) \\ e^{-t}\cos(2t) \end{array}\right),
\]

\[
x^{(2)} = \left(\begin{array}{c} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{array}\right).
\]

The general real-valued solution is

\[
x = C_1 \left(\begin{array}{c} -2e^{-t}\sin(2t) \\ e^{-t}\cos(2t) \end{array}\right) + C_2 \left(\begin{array}{c} 2e^{-t}\cos(2t) \\ e^{-t}\sin(2t) \end{array}\right).
\]

Here is a graph:
As you can see, the solutions spiral towards zero as \( t \to \infty \). So zero is what’s called an asymptotically stable spiral point.

Notice that we did not need to use both eigenvalues to find the complete set of real-valued solutions. In fact, the complex-valued solution given by an eigenvector for the second eigenvalue would just be the complex conjugate of the first complex-valued solution we found (or a scalar multiple thereof). So its real and imaginary part would give us no new information.

7.6.6. Express the solution of the given system of equations in terms of real-valued functions. Draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as \( t \to \infty \).

\[
\begin{bmatrix}
1 & 2 \\
-5 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

The characteristic polynomial is

\[
\begin{vmatrix}
1 - \lambda & 2 \\
-5 & -1 - \lambda
\end{vmatrix} = \lambda^2 + 9,
\]

whose roots are \( \pm 3i \). An eigenvector for the eigenvalue \( 3i \) is \( \begin{bmatrix} 1+3i \\ 5 \end{bmatrix} \). This gives us the complex-valued solution

\[
\begin{bmatrix}
1 + 3i \\
5
\end{bmatrix} e^{3it} = \begin{bmatrix}
(1 + 3i)(\cos(3t) + i\sin(3t)) \\
5(\cos(3t) + i\sin(3t))
\end{bmatrix} = \begin{bmatrix}
(\cos(3t) - 3\sin(3t)) + i(3\cos(3t) + \sin(3t)) \\
5\cos(3t) + 5i\sin(3t)
\end{bmatrix}.
\]

Taking the real and imaginary part of this, we get real-valued solutions

\[
\begin{bmatrix}
\cos(3t) - 3\sin(3t) \\
5\cos(3t)
\end{bmatrix}, \quad \begin{bmatrix}
3\cos(3t) + \sin(3t) \\
5\sin(3t)
\end{bmatrix}.
\]
The general real-valued solution is

\[
x = C_1 \begin{pmatrix} \cos(3t) - 3 \sin(3t) \\ 5 \cos(3t) \end{pmatrix} + C_2 \begin{pmatrix} 3 \cos(3t) + \sin(3t) \\ 5 \sin(3t) \end{pmatrix}.
\]

Here is a graph:

Solutions orbit around ellipses with \((0, 0)\) as the center.