# Week 14 solutions

ASSIGNMENT 34.

O. *Laplace transforms may be used to find solutions to some linear systems of differential equations. Consider the linear system of differential equations:*

$$\begin{cases} x' &= x + y \\ y' &= 4x + y \end{cases} \tag{1}$$

with initial conditions $x(0) = 0$ and $y(0) = 2$.

(a) *Let $X(s) = \mathcal{L}[x(t)]$ and $Y(s) = \mathcal{L}[y(t)]$ be the Laplace transforms of the functions $x(t)$ and $y(t)$, respectively. Take the Laplace transform of each of the differential equations in (**??**) and solve for $X(s)$ (i.e., eliminate $Y(s)$).*

If $X = \mathcal{L}[x]$ and $Y = \mathcal{L}[y]$, then using the initial conditions, $\mathcal{L}[x'] = sX$ and $\mathcal{L}[y'] = sY - 2$. So the Laplace transform of the system above is

$$\begin{cases} sX &= X + Y \\ sY - 2 &= 4X + Y \end{cases}$$

The first equation gives $Y = (s-1)X$. Substituting this into the second equation, we have

$$s(s-1)X - 2 = (s+3)X$$

or

$$X = \frac{2}{s^2 - 2s - 3} = \frac{2}{(s-3)(s+1)}.$$

(b) *Using the function $X(s)$ from (a), determine $x(t)$.*

First, we rewrite $X$ using partial fractions:

$$X = \frac{-1/2}{s+1} + \frac{1/2}{s-3}.$$

The inverse Laplace transform is

$$x = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{3t}.$$

(c) *Use the expression for $x(t)$ and the first equation in (**??**) to determine $y(t)$.*

We have

$$y = x' - x = \frac{1}{2}e^{-t} + \frac{3}{2}e^{3t} + \frac{1}{2}e^{-t} - \frac{1}{2}e^{3t} = e^{-t} + e^{3t}.$$

ASSIGNMENT 35.

P. *Find a particular solution* $\mathbf{x}_P(t)$ *of these nonhomogeneous systems:*

(a)

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 5e^{2t} \\ 3 \end{pmatrix}.$$

The 'forcing function' – the vector-valued function making the equation nonhomogeneous – is a linear combination of $e^{2t}$ and a constant function. The obvious thing to try for $\mathbf{x}$ for the method of undetermined coefficients is

$$\mathbf{x} = \mathbf{a}e^{2t} + \mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are unknown vectors. In order for this to work, we need to know that no solutions to the associated homogeneous equation are of this form – that is, that neither 2 nor 0 is an eigenvalue of the coefficient matrix. However, the characteristic polynomial of that coefficient matrix is just $(1-\lambda)(-3-\lambda)$, so the eigenvalues are 1 and $-3$. We don't actually need to find the general solution to the associated homogeneous equation – this is enough information to use the method of undetermined coefficients.

So let

$$\mathbf{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{2t} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Substituting $\mathbf{x}$ into the equation, we have

$$\begin{pmatrix} 2a_1 \\ 2a_2 \end{pmatrix} e^{2t} = \begin{pmatrix} a_1 \\ 2a_1 - 3a_2 \end{pmatrix} e^{2t} + \begin{pmatrix} b_1 \\ 2b_1 - 3b_2 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Comparing the coefficients of $e^{2t}$, we have

$$\begin{aligned} 2a_1 &= a_1 + 5 \\ 2a_2 &= 2a_1 - 3a_2. \end{aligned}$$

The solution is $\mathbf{a} = \binom{5}{2}$. Comparing the constant terms, we have

$$\begin{aligned} 0 &= b_1 \\ 0 &= 2b_1 - 3b_2 + 3 \end{aligned}$$

The solution is $\mathbf{b} = \binom{0}{1}$. So we finally get

$$\mathbf{x}_P(t) = \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b)

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 10\cos(t) \\ 0 \end{pmatrix}.$$

Again, since the coefficient matrix doesn't have $\pm i$ as eigenvalues, we should be able to find an $\mathbf{x}$ of the form

$$\mathbf{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cos(t) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sin(t).$$

Substituting this into the equation, we get

$$-\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sin(t) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cos(t) = \begin{pmatrix} a_1 \\ 2a_1 - 3a_2 \end{pmatrix} \cos(t) + \begin{pmatrix} b_1 \\ 2b_1 - 3b_2 \end{pmatrix} \sin(t) + \begin{pmatrix} 10 \\ 0 \end{pmatrix} \cos(t).$$

Comparing the coefficients of $\cos(t)$, we have

$$\begin{aligned} b_1 &= a_1 + 10 \\ b_2 &= 2a_1 - 3a_2. \end{aligned}$$

Comparing the coefficients of $\sin(t)$, we have

$$\begin{aligned} -a_1 &= b_1 \\ -a_2 &= 2b_1 - 3b_2. \end{aligned}$$

Unlike the previous problem, where these linear systems separated cleanly into two parts, in this problem we have to deal with all four equations at once. We get

$$a_1 = -5,\ b_1 = 5,\ a_2 = -4,\ b_2 = 2.$$

So a solution is

$$\mathbf{x}_P(t) = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \cos(t) + \begin{pmatrix} 5 \\ 2 \end{pmatrix} \sin(t).$$