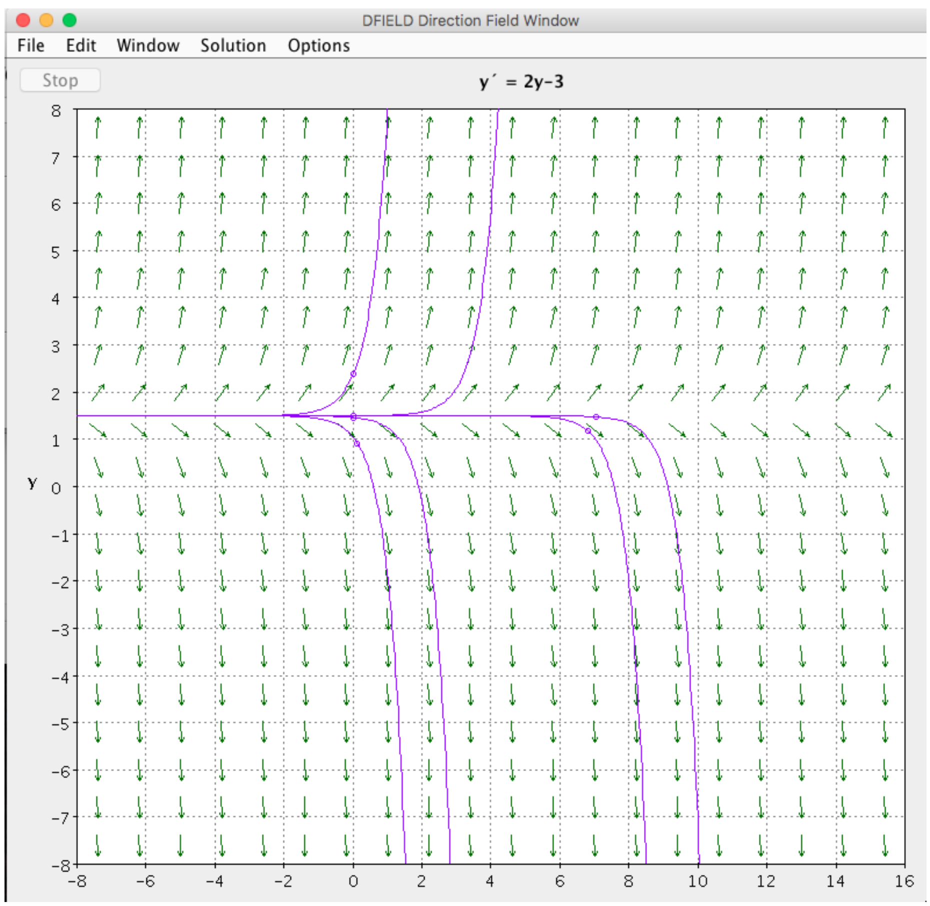
Solutions to Week 1 Homework

ASSIGNMENT 1.

1.1.2. *Draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.*

$$y' = 2y - 3$$

I've used a sample output from the dfield program – you could have used this or just sketched the direction field by hand.



As you can see, solutions with high initial values $y(0)$ go to ∞ as $t \to \infty$, and solutions with low initial vaues go to $-\infty$. In between, there is a critical value $y(0) = 1.5$ where the solution is a horizontal line. Thus, the limiting behavior is

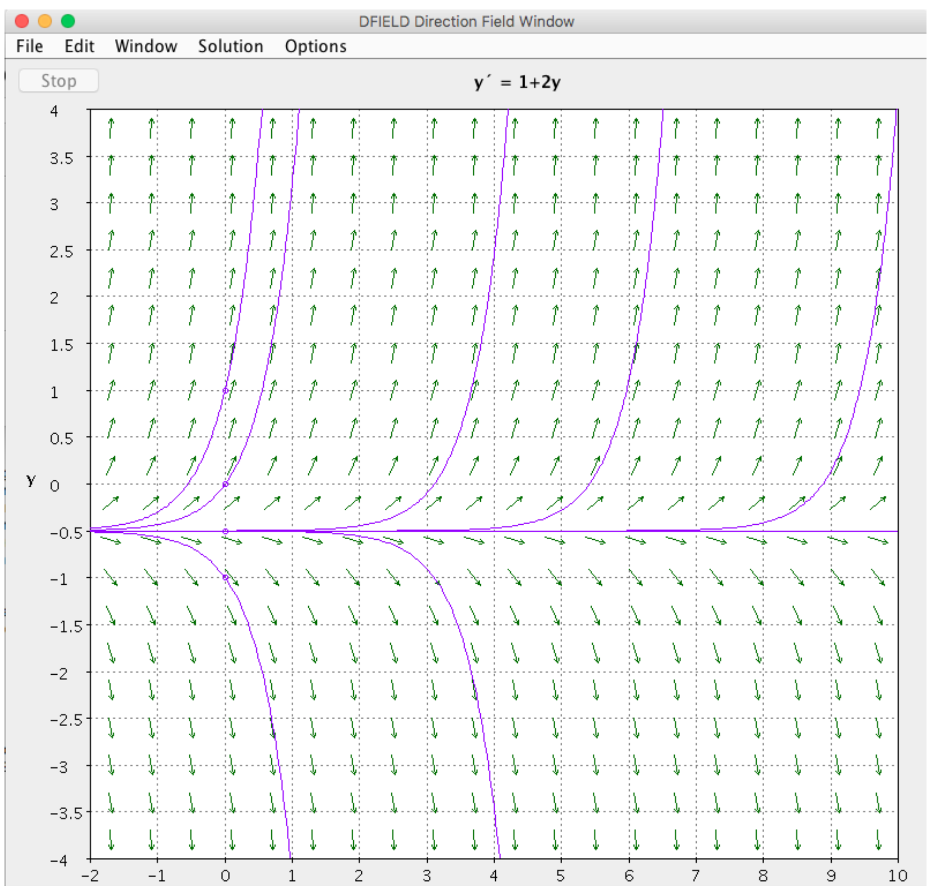
$$\lim_{t\to\infty} y = \begin{cases} y(0) > 1.5 & y \to \infty \\ y(0) = 1.5 & y \to 1.5 \\ y(0) < 1.5 & y \to -\infty \end{cases}.$$

If you found solutions by clicking around on the dfield screen, you might not have noticed the critical solution. In fact, in my screenshot, none of the arrows are actually horizontal. How would you figure out that it's there? First, you could've guessed that there was some sort of transition between the solutions that diverge to ∞ and those that diverge to $-\infty$. If this transitional solution is supposed to look like a horizontal line, then y' should be 0 at any point along the solution. Now, looking at the equation, we see that $y' = 0$ just when $y = 1.5$. So the constant function $y = 1.5$ is a solution to the equation.

1.1.5. *Draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.*

$$y' = 1 + 2y$$

Here's a picture.



By similar reasoning as in the previous problem, the limiting behavior is

$$\lim_{t\to\infty} y = \begin{cases} y(0) > -0.5 & y \to \infty \\ y(0) = -0.5 & y \to -0.5 \\ y(0) < -0.5 & y \to -\infty \end{cases}.$$

At the critical value $y(0) = -0.5$, the solution is a constant function $y = -0.5$.

Remark. 'Find the limiting behavior' means at the very least 'find the limit', and for these problems, that's enough. Sometimes, you might want to say more. Does the function oscillate as it approaches its limit? If it diverges to ∞, does it do so exponentially, linearly, quadratically, ...? Is there anything else interesting you notice?

ASSIGNMENT 2.

1.3.8. *Verify that each given function is a solution of the differential equation.*

$$y'' + 2y' - 3y = 0, \quad y_1(t) = e^{-3t}, \quad y_2(t) = e^t.$$

First consider y_1. We have $y_1'(t) = -3e^{-3t}$, and $y_1''(t) = 9e^{-3t}$. Thus,

$$y_1'' + 2y_1' - 3y_1 = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0.$$

For y_2, we have $y_2'(t) = y_2''(t) = e^t$. Thus,

$$y_2'' + 2y_2' - 3y_2 = e^t + 2e^t - 3e^t = 0.$$

1.3.11. *Verify that each given function is a solution of the differential equation.*

$$2t^2y'' + 3ty' - y = 0, \ t > 0; \quad y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1}.$$

For y_1, we have $y_1'(t) = (1/2)t^{-1/2}$ and $y_1''(t) = (-1/4)t^{-3/2}$. Thus,

$$2t^2y_1'' + 3ty_1' - y_1 = 2\left(-\frac{1}{4}\right)t^2 \cdot t^{-3/2} + 3 \cdot \frac{1}{2}t \cdot t^{-1/2} - t^{1/2} = -\frac{1}{2}t^{1/2} + \frac{3}{2}t^{1/2} - t^{1/2} = 0.$$

We used the rule for multiplying exponents with the same base: $t^a \cdot t^b = t^{a+b}$.

For y_2, we have $y_2'(t) = -t^{-2}$ and $y_2''(t) = 2t^{-3}$. We calculate

$$2t^2y_2'' + 3ty_2' - y_2 = 4t^2 \cdot t^{-3} - 3t \cdot t^{-2} - t^{-1} = 4t^{-1} - 3t^{-1} - t^{-1} = 0.$$

A. *For what values of A, if any, will Ate^{-2t} be a solution of the differential equation $2y' + 4y = 3e^{-2t}$? For what values of B, if any, will $y = Be^{-2t}$ be a solution?*

We can plug in these solutions into the equation. If $y = Ate^{-2t}$, then

$$y' = -2Ate^{-2t} + Ae^{-2t}.$$

Therefore,

$$2y' + 4y = -4Ate^{-2t} + 2Ae^{-2t} + 4Ate^{-2t} = 2Ae^{-2t}.$$

Thus, this is only a solution for $A = 3/2$.

If $y = Be^{-2t}$, then $y' = -2Be^{-2t}$. We get

$$2y' + 4y = -4Be^{-2t} + 4Be^{-2t} = 0,$$

so this is never a solution.

Remark. At this point, you should be able to find all the solutions to this equation, using the integrating factor method. The integrating factor is $\mu(t) = e^{2t}$, and we get

$$2e^{2t}y' + 4e^{2t}y = 3,$$

which integrates to give

$$2e^{2t}y = 3t + C,$$

and

$$y = \frac{3}{2}te^{-2t} + Ce^{-2t}$$

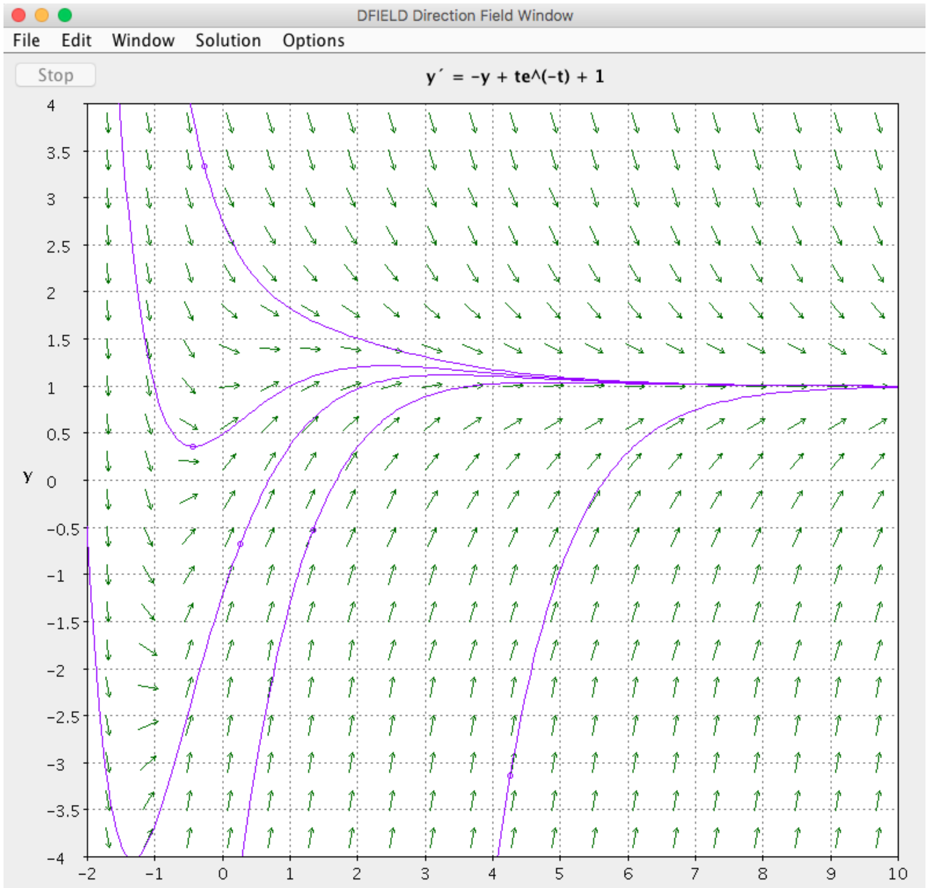
as the general solution.

ASSIGNMENT 3.

2.1.3. (a) *Draw a direction field for the given differential equation.*

$$y' + y = te^{-t} + 1$$

Here's a picture from dfield:



(b) *Based on an inspection of the direction field, describe how solutions behave for large t.*

The solutions all converge asymptotically to $y = 1$ as $t \to \infty$.

(c) *Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.*

We can solve the equation using the integrating factor method. Multiplying by the integrating factor $\mu(t)$, we have

$$\mu y' + \mu y = \mu \cdot (te^{-t} + 1).$$

The left-hand side should be equal to $\frac{d}{dt}(\mu y) = \mu y' + \mu' y$. Thus, $\mu' = \mu$. One such function is $\mu(t) = e^t$. Using this, we have

$$\frac{d}{dt}(e^t y) = e^t(te^{-t} + 1) = t + e^t.$$

Integrating gives

$$e^t y = e^t + \frac{t^2}{2} + C,$$

so

$$y = 1 + \frac{t^2 e^{-t}}{2} + Ce^{-t}.$$

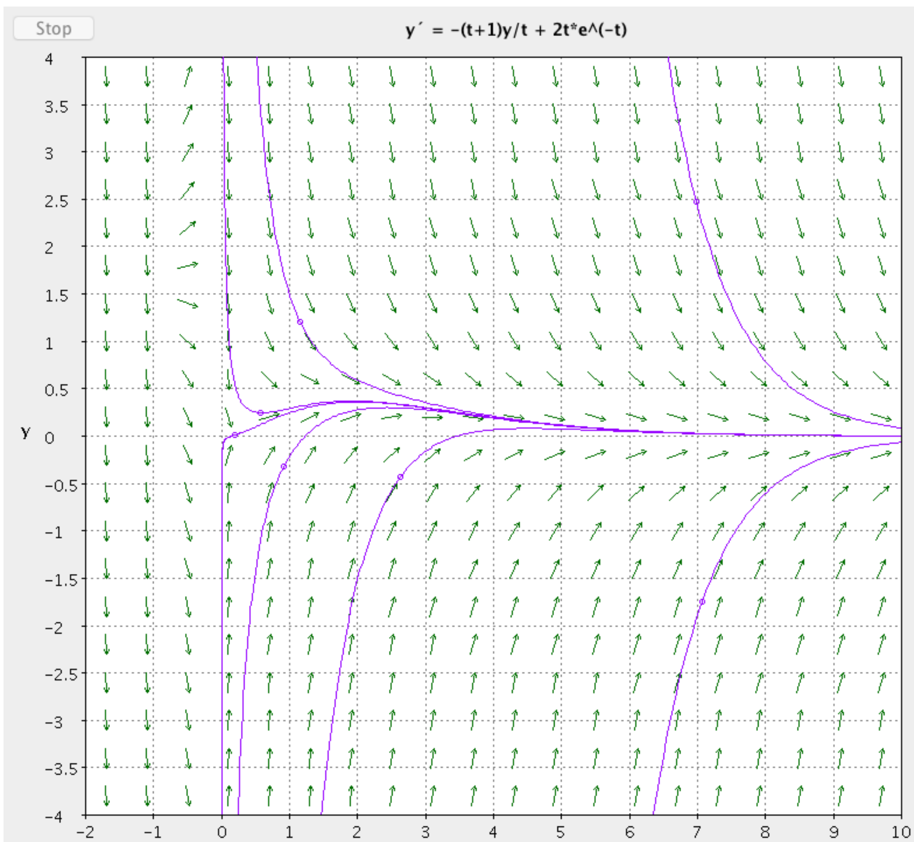
For any value of C,

$$\lim_{t\to\infty} y = 1.$$

2.1.24. (a) *Draw a direction field for the given differential equation. How do solutions appear to behave as $t \to 0$? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.*

$$ty' + (t+1)y = 2te^{-t}, \; y(1) = a \; t > 0$$

Here is a picture:



We can see two types of solutions. For high values of a, the solutions diverge to $+\infty$ as $t \to 0$. For low values of a, the solutions diverge to $-\infty$. The critical value a_0 appears to be around 0.25.

(b) *Solve the initial value problem and find the critical value a_0 exactly.*

Multiplying by an integrating factor μ gives

$$t\mu y' + (t+1)\mu y = 2t\mu e^{-t}.$$

We want

$$\frac{d}{dt}(t\mu) = (t+1)\mu.$$

But

$$\frac{d}{dt}(t\mu) = \mu + t\mu',$$

so we have $\mu' = \mu$. Again, $\mu = e^t$ works. (It's also possible to divide by t first, in which case you'd find the integrating factor te^t.) The differential equation becomes

$$te^t y' + (t+1)e^t y = 2t.$$

Integrating gives

$$te^t y = t^2 + C.$$

Thus,

$$y = te^{-t} + Ct^{-1}e^{-t}.$$

The initial value is

$$a = y(1) = e^{-1} + Ce^{-1} = (1+C)e^{-1}.$$

So

$$C = ea - 1,$$

putting the equation in the form

$$y = te^{-t} + (ea-1)t^{-1}e^{-t}.$$

As $t \to 0$, the e^{-t} factors go to 1 and the first term vanishes. So

$$\lim_{t\to 0} y = \lim_{t\to 0}(ea-1)t^{-1}.$$

Clearly, the sign of t^{-1} is the thing that matters here. The critical value satisfies $ea_0 - 1 = 0$, or $a_0 = 1/e$.

(c) *Describe the behavior of the solution corresponding to the initial value a_0.* When $a = a_0 = 1/e$, the solution is

$$y = te^{-t}.$$

This satisfies $y(0) = 0$. Unlike all the other solutions, this one actually extends continuously to negative values of t.