# Solutions to Week 2 Homework

**ASSIGNMENT 4.**

**2.2.11.**

$$x\,dx + ye^{-x}\,dy = 0, \quad y(0) = 1$$

(a) *Find the solution of the given initial value problem in explicit form.*

Multiplying through by $e^x$ separates the variables:

$$xe^x\,dx + y\,dy = 0.$$

We can now integrate. We can integrate $xe^x\,dx$ by parts, with $u = x$ and $dv = e^x\,dx$. We have $du = dx$ and $v = e^x$. Integration by parts gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

The equation turns into

$$xe^x - e^x + y^2/2 = C.$$

If we want $y$ as a function of $x$, we rewrite this as

$$y = \pm\sqrt{2e^x - 2xe^x + C}.$$

Plugging in $y(0) = 1$ gives

$$1 = \pm\sqrt{2 + C},$$

so that $C = -1$. We obtain

$$y = \pm\sqrt{2e^x - 2xe^x - 1} \tag{1}$$

or

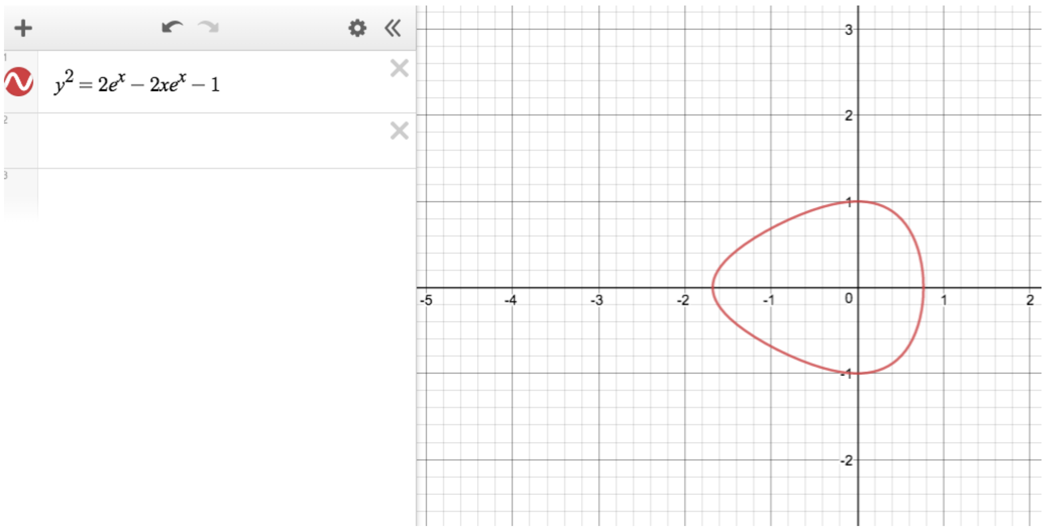
$$y^2 = 2e^x - 2xe^x - 1. \tag{2}$$

Note that either the positive or the negative square root works. Either of the equations (1) or (2) is valid – the first expresses $y$ as one of two functions of $x$, the second expresses a relation between $y$ and $x$. It would not be valid to only write

$$y = \sqrt{2e^x - 2xe^x - 1}.$$

This would be akin to giving only one of two solutions to a quadratic equation.

(b) *Plot the graph of the solution.*




Again, there are two ways of thinking about this: either it's a graph of a non-functional relation between $x$ and $y$, or it's two possible solutions of $y$ in terms of $x$.

(c) *Determine (at least approximately) the interval in which the solution is defined.*

As we see from the graph, both solutions are defined on some finite, open interval. Since the problem asks for an approximate solution, it's fine to just look at the graph to get an answer. The solution is defined approximately on the interval $[-1.678, 0.768]$.

**2.2.14.**

$$y' = xy^3(1+x^2)^{-1/2}, \quad y(0) = 1$$

(a) *Find the solution of the given initial value problem in explicit form.*

First, separate the variables:

$$y^{-3}\,dy = \frac{x}{\sqrt{1+x^2}^{1/2}}\,dx.$$

The integral of the left-hand side is $\frac{y^{-2}}{-2} + C$. There are a few ways to integrate the right-hand side. One is to make the substitution

$$u = x^2, \quad du = 2x\,dx.$$

Then

$$\int \frac{x}{\sqrt{1+x^2}^{1/2}}\,dx = \int \frac{1}{2\sqrt{1+u}}\,du = \sqrt{1+u} + C = \sqrt{1+x^2} + C.$$

We finally obtain

$$-\frac{y^{-2}}{2} = -\frac{1}{2y^2} = \sqrt{1+x^2} + C.$$

which simplifies to

$$y^2 = \frac{1}{C - 2\sqrt{1+x^2}} \tag{3}$$

or

$$y = \pm(C - 2\sqrt{1+x^2})^{-1/2}. \tag{4}$$

(Again, $y$ can be either of the two square roots – either one solves the differential equation. On the other hand, only the positive square root of $1+x^2$ appears inside. This is because $(1+x^2)^{-1/2}$ was in the original differential equation, implicitly meaning the positive square root.)

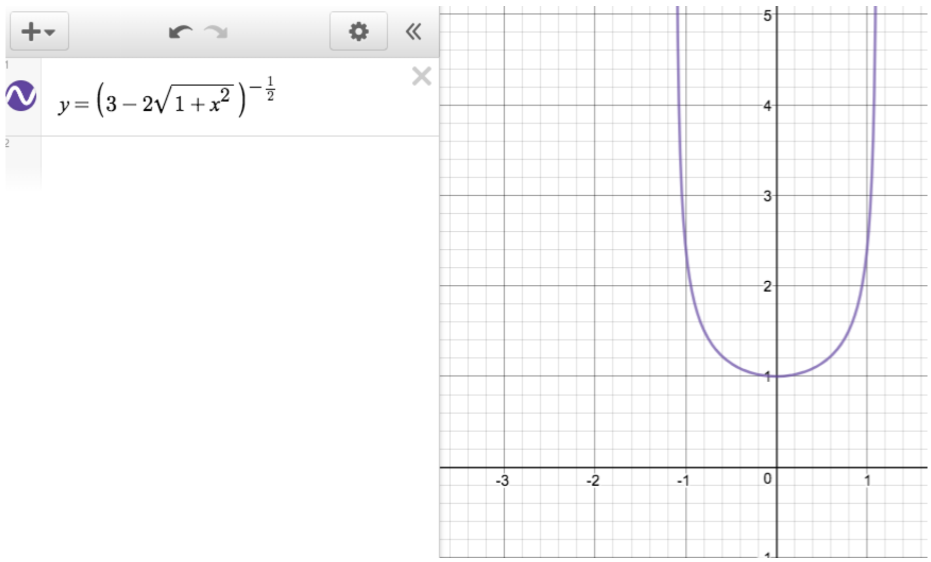
Finally, plugging in $y(0) = 1$ gives

$$1 = \pm(C-2)^{-1/2},$$

so that the positive square root is required and $C = 3$. The solution to the initial value problem is

$$y = (3 - 2\sqrt{1+x^2})^{-1/2}. \tag{5}$$

(b) *Plot the graph of the solution.*




(c) *Determine (at least approximately) the interval in which the solution is defined.*
The solution appears to have vertical asymptotes at $x = \pm 1.1$. We can check this explicitly by looking at the equation (5). This $y$ is only undefined where

$$3 - 2\sqrt{1+x^2} \leq 0,$$

or in other words, where

$$1 + x^2 \geq 9/4.$$

or

$$x \leq -\sqrt{5}/2 \text{ or } x \geq \sqrt{5}/2 \approx 1.118.$$

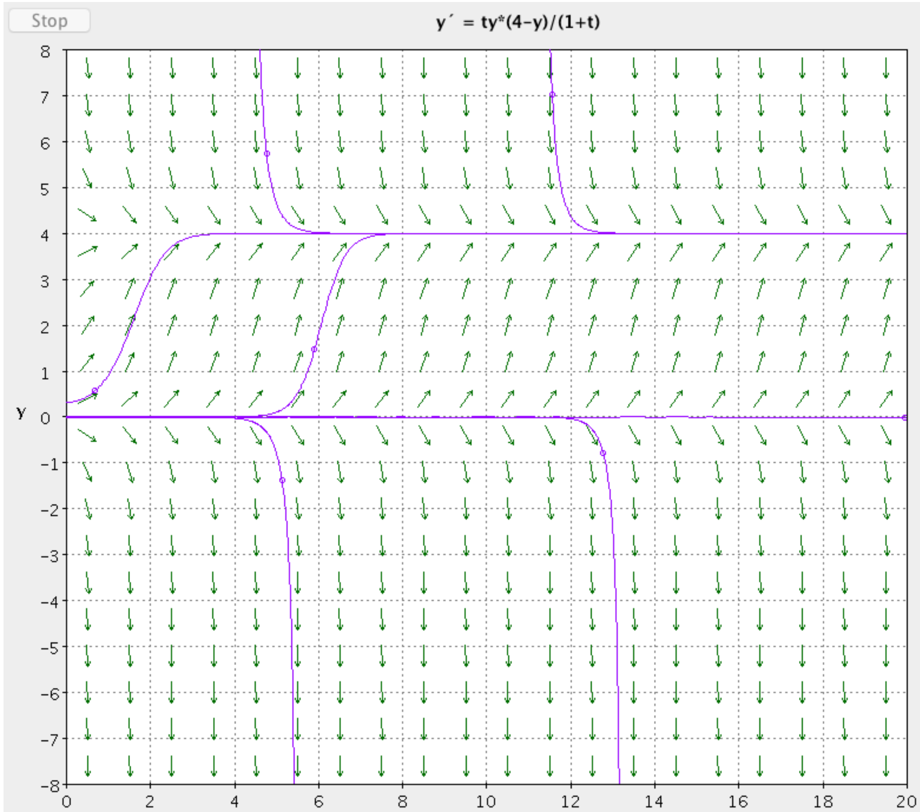
Thus, the interval of definition of $y$ is indeed $(-1.118, 1.118)$.

**2.2.28.** *Consider the initial value problem*

$$y' = ty(4-y)/(1+t), \quad y(0) = y_0 > 0.$$

I will approach this problem two ways: by looking at graphs, and by solving the equation explicitly.

(a) *Determine how the solution behaves as $t \to \infty$.*
**Graphically:** Here is a graph of some solutions to the equation.




We can see two critical (or 'equilibrium') solutions, at $y = 0$ and 4. In fact, it's easy to see from the differential equation that these are solutions: $y'$ is identically zero along these lines. Solutions with initial values $0 < y_0 < 4$ approach 4 asymptotically, as do solutions with $y_0 > 4$. (This means that 4 is an 'asymptotically stable equilibrium' – nearby solutions approach this one over time – and 0 is an 'unstable equilibrium.')

(When I first graphed this, 4 was the upper limit on my graphing window, and I only saw the critical solution $y = 0$. I noticed to zoom out when I went back to the equation to check that 0 was actually a solution, and noticed that 4 was also a zero of $y'$. This shows how the graph and the algebra can play off each other in helping you understand equations.)

**Explicitly:** The equation is separable, so we separate it:

$$\frac{1}{y(4-y)}\,dy = \frac{t}{1+t}\,dt.$$

The right-hand side can be expanded to $\left(1 - \frac{1}{1+t}\right)dt$, with integral $t - \ln(1+t) + C$. We expand the left-hand side using partial fractions. If

$$\frac{1}{y(4-y)} = \frac{A}{y} + \frac{B}{4-y},$$

then

$$1 = A(4-y) + By = 4A + (B-A)y.$$

Thus, $A = B = 1/4$. We have

$$\int \frac{1}{y(4-y)}\,dy = \frac{1}{4}\int\left(\frac{1}{y} + \frac{1}{4-y}\right)dy = \frac{1}{4}\left(\ln|y| - \ln|4-y|\right) + C.$$

Thus, the whole equation integrates to

$$\ln|y| - \ln|4-y| = 4t - 4\ln(1+t) + C.$$

Now exponentiating, we get

$$\frac{|y|}{|4-y|} = A\frac{e^{4t}}{(1+t)^4} \tag{6}$$

where $A$ is a constant of integration. (If you're uncomfortable with exponentials and logarithms, it's a good exercise to check this last step carefully.)

Although we're not done solving the equation, we have enough to answer the question. First, note that both the left-hand side of (6) and $\frac{e^{4t}}{(1+t)^4}$ are always nonnegative, so $A \geq 0$. If $A > 0$, then right-hand side of (6) goes to $\infty$ as $t \to \infty$, so $y$ must go to 4. We would also notice at this point that $y = 4$ is a solution to the equation, though it does not appear as a special case of (6). We lost track of it all the way in the beginning, when we divided by $y(4-y)$! If $A = 0$, then $y$ is identically 0 – but this doesn't count, because we only care about solutions with $y(0) > 0$.

Just for the heck of it, let's finish solving for $y$. Now it helps to let the graph inform the algebra! We see from the graph that solutions with $y_0 > 0$ stay positive, and if $y_0 < 4$, then $y(t) < 4$ for all $t$. So the absolute values split into two cases. If $y_0 < 4$, then (6) turns into

$$\begin{aligned}
\frac{y}{4-y} &= A(1+t)^{-4}e^{4t} \\
y &= (4-y)A(1+t)^{-4}e^{4t} \\
y &= \frac{4A(1+t)^{-4}e^{4t}}{1+A(1+t)^{-4}e^{4t}} \\
y &= \frac{4Ae^{4t}}{(1+t)^4 + Ae^{4t}}.
\end{aligned}$$

The other case, where $y > 4$, gives the same answer as this case after replacing $A$ with $-A$.

(b) *If $y_0 = 2$, find the time $T$ at which the solution first reaches the value 3.99.*

We have

$$y_0 = \frac{4A}{1+A} = 2,$$

whose solution is $A = 1$. We want to solve

$$\frac{3.99}{0.01} = 399 = (t+1)^{-4}e^{4t}.$$

It is probably best to use a calculator or computer to do this. I did it by graphing the curve in Desmos and zooming in, and got $t = 2.844$.

(c) *Find the range of initial values for which the solution lies in the interval* $3.99 < y < 4.01$ *by the time* $t = 2$.

First, let's think about the geometry of the problem. When $y_0 < 4$, the solution approaches 4 asymptotically from the bottom, and smaller positive values of $y_0$ approach 4 more slowly. When $y_0 > 4$, the solution approaches 4 asymptotically from the top, and larger values of $y_0$ approach 4 more slowly. So we should expect to find a single interval of initial values, clustered around 4. The endpoints of the interval are just those $y_0$'s for which $y(2) = 3.99$ or 4.01.

We rewrite our general solution (6). Since

$$y_0 = \frac{4A}{1+A},$$

we have

$$A = \frac{y_0}{4 - y_0}.$$

So in terms of $y_0$, the general solution is

$$y = \frac{4y_0 e^{4t}}{(4 - y_0)(1+t)^4 + y_0 e^{4t}} \tag{7}$$

Now,

$$y(2) = \frac{4y_0 e^8}{(4 - y_0) \cdot 3^4 + y_0 e^8}.$$

We solve for $y_0$ in terms of $y(2)$.

$$\begin{aligned} \left[81(4 - y_0) + e^8 y_0\right] \cdot y(2) &= 4e^8 y_0 \\ 4e^8 y_0 - e^8 y(2) y_0 + 81 y(2) y_0 &= 4 \cdot 81 \cdot y(2) = 324 y(2) \\ y_0 &= \frac{324 y(2)}{4e^8 - e^8 y(2) + 81 y(2)}. \end{aligned}$$

When $y(2) = 3.99$, $y_0 \approx 3.662$. When $y(2) = 4.01$, $y_0 \approx 4.404$. Thus, we need $3.662 < y_0 < 4.4404$.

**ASSIGNMENT 5.**

**2.2.36.**

$$(x^2 + 3xy + y^2)\, dx - x^2\, dy = 0.$$

(a) *Show that the given differential equation is homogeneous.* We recall that a homogeneous equation is one that can be written in the form

$$dy/dx = f(y/x).$$

Here, we can simplify the equation to

$$dy/dx = (x^2 + 3xy + y^2)/(x^2) = 1 + 3(y/x) + (y/x)^2.$$

(b) *Solve the differential equation.* As we discussed in class (this is also covered in the text preceding this problem), the substitution $v = y/x$ is often helpful for solving homogeneous equations. Since $y = vx$, the product rule gives

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

So the equation put in terms of $v$ and $x$ is

$$v + x\frac{dv}{dx} = 1 + 3v + v^2.$$

We can now separate the variables (which was the point of doing this substitution):

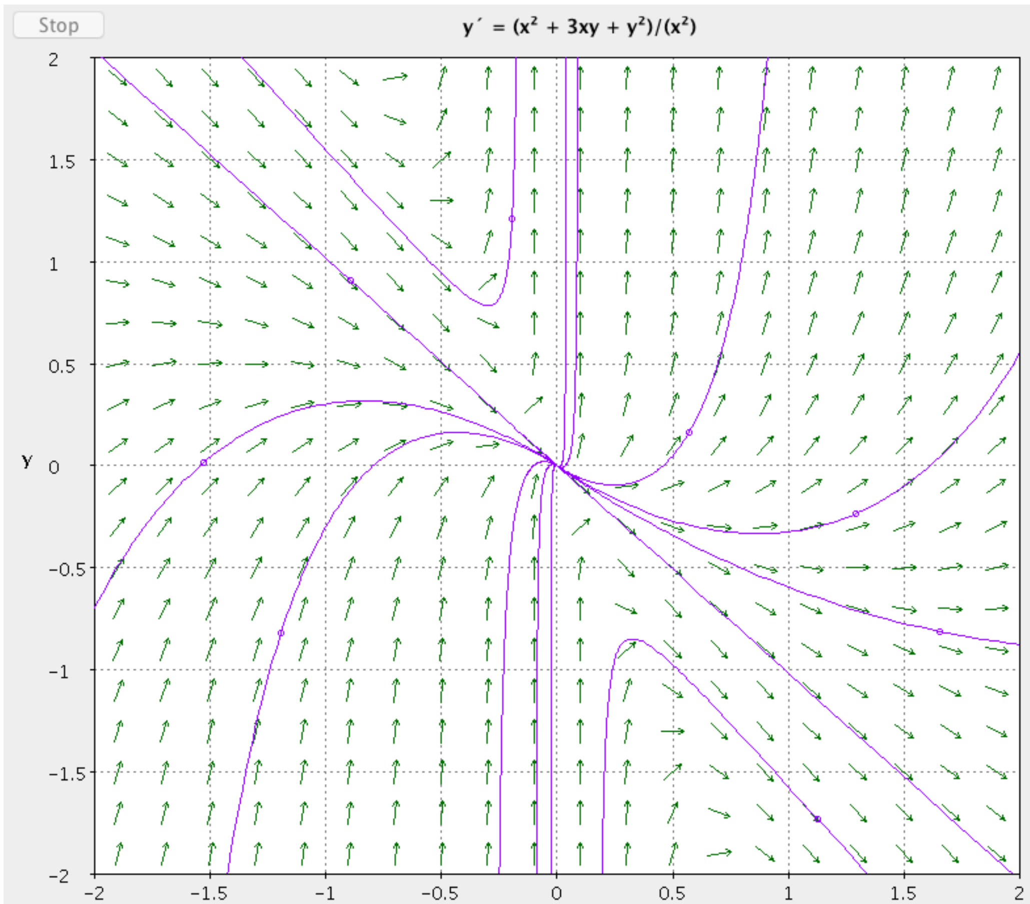
$$\begin{aligned} x\frac{dv}{dx} &= 1 + 2v + v^2 \\ \frac{1}{1 + 2v + v^2}\, dv &= \frac{1}{x}\, dx \\ \int \frac{1}{(v+1)^2}\, dv &= \int \frac{1}{x}\, dx \\ -(v+1)^{-1} &= \ln|x| + C \\ v + 1 &= \frac{-1}{\ln|x| + C} \\ v &= \frac{-1}{\ln|x| + C} - 1. \end{aligned}$$

To solve the equation, we need to put $y$ in terms of $x$. Since $v = y/x$, we ultimately obtain

$$y = \frac{-x}{\ln|x| + C} - x. \tag{8}$$

Since we care about negative values of $x$ in the next part of the problem, we do need the absolute value here.

(c) *Draw a direction field and some integral curves. Are they symmetric with respect to the origin?*




As you can see, the integral curves are symmetric with respect to the origin. dfield can get confused about what happens when the integral curves pass through the origin, since $dy/dx$ is not defined there. However, one can also observe from (8) that, for a given value of $C$ and any value $a$ of $x$, $y(-a) = -y(a)$.

On the graph, one should also notice the linear solution $y = -x$. We did not obtain this in part (b) above, but checking it against the original equation shows that it is a solution. In fact, it appears to correspond to "$C = \infty$" in (8). What's going on? If you didn't find this solution, try to look at the derivation of (8) and see where this solution was forgotten about before continuing.

Here's the answer: $y = -x$ is a constant solution in terms of $v$, $v = -1$. Since this is a zero of $1 + 2v + v^2$, it was omitted when we divided by $(1 + v)^2$. It's good practice to pay attention to these divisions, especially when you separate variables, and see if there are any special solutions that would result in you dividing by zero.

**B.** *Using the substitution $u(x) = y + x$, solve the differential equation*

$$\frac{dy}{dx} = (y+x)^2.$$

If $u = y + x$, then $du/dx = dy/dx + 1$, or $dy/dx = du/dx - 1$. In terms of $u$ and $x$, the differential equation is

$$\frac{du}{dx} - 1 = u^2.$$

Separating variables gives

$$\frac{1}{1+u^2}\,du = dx,$$

which integrates to

$$\arctan(u) = x + C$$

or

$$u = \tan(x + C).$$

Putting this back in terms of $y$ gives

$$y = \tan(x + C) - x.$$

One noticeable thing about this differential equation is that the interval of validity of a given solution depends on $C$. For a given $C$, the solution has vertical asymptotes at $\pm\pi/2 - C$.

**C.** *Using the substitution $u(x) = y^3$, solve the differential equation*

$$y^2\frac{dy}{dx} + \frac{y^3}{x} = \frac{2}{x^2}, \quad x > 0.$$

Since $u = y^3$, $du/dx = 3y^2(dy/dx)$, so

$$\frac{dy}{dx} = \frac{1}{3y^2}\frac{du}{dx}.$$

Substituting gives

$$\frac{1}{3}\frac{du}{dx} + \frac{u}{x} = \frac{2}{x^2},$$

which is an equation in terms of $x$ and $u$. This is a linear equation, which we can solve using the integrating factor method. Since it has the form

$$u' + 3u/x = 6/x^2,$$

the integrating factor is

$$e^{\int (3/x)\,dx} = e^{3\ln x} = x^3.$$

We calculate

$$\begin{aligned}
x^3u' + 3x^2u &= 6x \\
\frac{d}{dx}(x^3u) &= 6x \\
x^3u &= \int 6x\,dx \\
x^3u &= 3x^2 + C \\
u &= 3x^{-1} + Cx^{-3} \\
y^3 &= 3x^{-1} + Cx^{-3} \\
y &= \sqrt[3]{3x^{-1} + Cx^{-3}}.
\end{aligned}$$

**ASSIGNMENT 6.**

**2.3.1.** *Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches* 1% *of its original value.*

Let $Q$ be the amount of dye in the tank. Every minute, 2 liters of the tank water – which is 1/100th of the total water in the tank – is replaced by 2 liters of fresh water. So we have the differential equation

$$\frac{dQ}{dt} = -Q/100.$$

This is a classic exponential decay equation. The general solution is

$$Q(t) = Ae^{-t/100}.$$

The initial value is $Q(0) = 200\,\mathrm{g}$, so the specific solution is

$$Q(t) = 200e^{-t/100}\ [\mathrm{g}].$$

Now, we want to find the $t$ at which $Q(t) = 2\,\mathrm{g}$. So

$$\begin{aligned} 1/100 &= e^{-t/100} \\ \ln(1/100) &= -t/100 \\ t &= -100\ln(1/100) = 460.6\,\mathrm{min}. \end{aligned}$$

One neat feature of this problem, since it's an exponential decay, is that the time it takes for 1% of the original amount of dye to be left doesn't depend on the initial amount of dye. We could have left 200 g as $A$ and found the same answer.

**2.3.2.** *A tank initially contains 120 L of pure water. A mixture containing a concentration of $\gamma$ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of $\gamma$ for the amount of salt in the tank at any time $t$. Also find the limiting amount of salt in the tank as $t \to \infty$.*

If the amount of salt in the tank is $Q$, then every minute, 1/60th of the tank's water leaves the tank, so $Q/60$ salt leaves the tank. We also have $2\gamma$ liters of salt entering the tank every minute. We get the differential equation

$$\frac{dQ}{dt} = 2\gamma - \frac{Q}{60}.$$

We separate the variables:

$$\frac{1}{Q - 120\gamma}\,dQ = \frac{-1}{60}\,dt$$

and integrate to get

$$\ln|Q - 120\gamma| = -t/60 + C.$$

The solution is

$$Q = 120\gamma + Ae^{-t/60}.$$

If the initial value is $Q(0) = 0$, then the specific solution is

$$Q = 120\gamma - 120\gamma e^{-t/60}.$$

As $t \to \infty$, the second term vanishes and we get

$$\lim_{t\to\infty} Q = 120\gamma.$$

This makes sense: the concentration of salt in the tank, $Q/120$, approaches that in the input mixture.