Solutions to Week 3 Homework

ASSIGNMENT 7.

2.3.16. *Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of* 200° F *when freshly poured, and 1 min later has cooled to* $190^\circ F$ *in a room at* 70° F*, determine when the coffee reaches a temperature of* $150^\circ F$. [5 pts]

Let T be the temperature of the object, and T_s the temperature of the surroundings. We can write Newton's law of cooling in equation form as follows:

$$\frac{dT}{dt} = -k(T - T_s). \tag{1}$$

Here k is a still-unknown constant, greater than zero. Let's double-check the sign: if the object has a higher temperature than its surroundings, then $T > T_s$, so dT/dt is negative, so the object is cooling, which is what we expect.

We solve the equation by separating the variables.

$$\begin{aligned}
\int \frac{1}{T - T_s}\, dT &= \int -k\, dt \\
\ln|T - T_s| &= -kt + C \\
|T - T_s| &= Ae^{-kt} \\
T - T_s &= Ae^{-kt} \quad \text{(replacing } A \text{ with } \pm A) \\
T &= Ae^{-kt} + T_s
\end{aligned}$$

In the given problem (with units degrees Fahrenheit), we have $T_s = 70$ and $T(0) = 200$, so we get $A = 130$. Since $T(1) = 190$,

$$\begin{aligned}
190 &= 130e^{-k} + 70 \\
120 &= 130e^{-k} \\
120/130 &= e^{-k} \\
k &= \ln(120/130) \approx 0.08
\end{aligned}$$

Finally, we solve for t in $T(t) = 150$.

$$\begin{aligned}
150 &= 130e^{-0.08t} + 70 \\
80 &= 130e^{-0.08t} \\
80/130 &= e^{-0.08t} \\
\ln(80/130) &= -0.08t \\
t &= 6.066 \text{ min.}
\end{aligned}$$

21. *A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. There is a force due to air resistance of magnitude* $|v|/30$ *directed opposite to the velocity, where the velocity* v *is measured in m/s.* [5 pts]

(a) *Find the maximum height above the ground that the ball reaches.* First we set up a differential equation. The right model here is

$$F = -mg - v/30$$

or

$$v' = -g - \frac{v}{30m} \tag{2}$$

Again, check the signs. $-g = -9.8\text{m/s}^2$ is negative, so the force of gravity pulls the ball down, as expected. Meanwhile, the term $-v/30m$ is oriented opposite to the velocity – it's positive if v is negative (the ball is going down), and negative if v is positive (the ball is going up). This is how we expect drag to behave. Notice also that the drag *force* has magnitude $|v|/30$, so that *acceleration* due to drag has magnitude $|v|/30m$.

Let's solve the equation.

$$\begin{aligned}
\int \frac{1}{v + 30mg}\, dv &= \int \frac{-1}{30m}\, dt \\
\ln|v + 30mg| &= -t/30m + C \\
|v + 30mg| &= Ae^{-t/30m} \\
v + 30mg &= Ae^{-t/30m} \quad \text{(replacing } A \text{ with } \pm A) \\
v &= Ae^{-t/30m} - 30mg \\
v &= Ae^{-t/4.5} - 44.1.
\end{aligned}$$

Since $v(0) = 20$, we must have $A = 64.1$:

$$v = 64.1e^{-t/4.5} - 44.1. \tag{3}$$

Now, the ball reaches its maximum height when $v = 0$ (and assuming the graph of the ball's height over time is roughly a parabola, this should be the only time that $v = 0$ (except when it lands on the ground, which isn't described by our model)). Solving for t gives

$$t = 1.683\,\text{s}.$$

Unfortunately, this doesn't answer the question – we're looking for the maximum height, not the time it reaches that height.

We need a formula for the height x, so we integrate again.

$$\begin{aligned}
x &= \int (64.1e^{-t/4.5} - 44.1)\, dt \\
&= -288.45e^{-t/4.5} - 44.1t + C \\
&= -288.45e^{-t/4.5} - 44.1t + 318.45 \quad (\text{since } x(0) = 30)
\end{aligned}$$

When $t = 1.863$ s,

$$x = 45.783\,\text{m}.$$

Remark. I like to keep track of units, particularly as a way of checking against silly math mistakes. Since velocity is in m/s but force is in kg $\cdot$ m/s^2, the 30 appearing in the expression 'drag force $= -v/30$' must be in s/kg. Then in $e^{-t/30m}$, the exponent ends up being dimensionless. In physics, this should happen with anything you put as an exponent – it makes sense to square units or to multiply different units together, but e to the power of seconds would be kind of weird. (As far as I know!)

(b) *Find the time that the ball hits the ground.*

This happens when $x = 0$, so we need to solve

$$-288.45e^{-t/4.5} - 44.1t + 318.45 = 0.$$

You can presumably do this on fancy graphing calculators, or using MATLAB or the like. I plotted the curve in Desmos and found that it hits the x-axis at

$$t = 5.129\,\mathrm{s}.$$

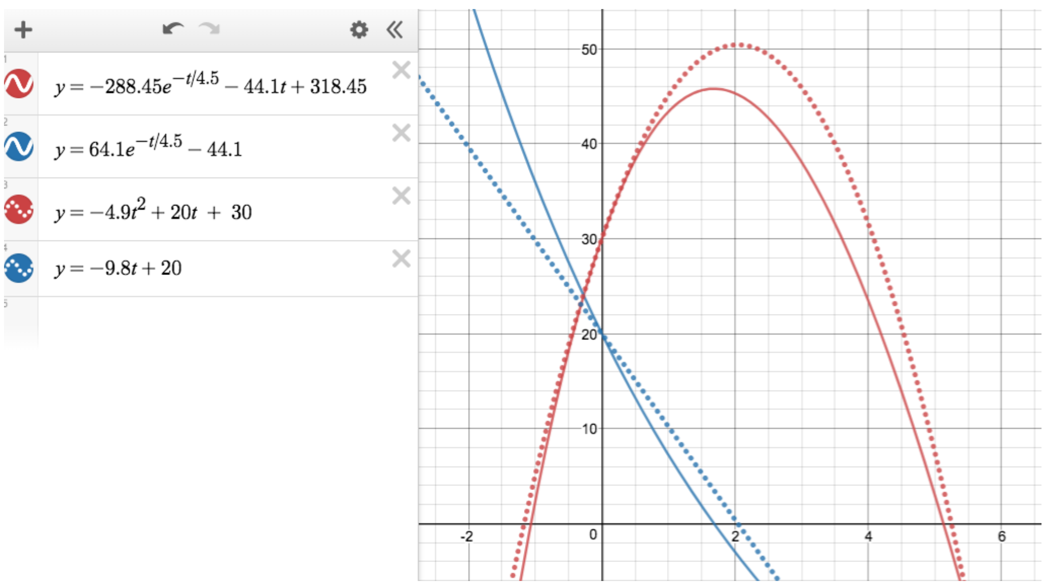
(c) *Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones of problem 20.*

You had to do problem 20 on WebAssign, but possibly with different numbers. In any case, it's the same setup without air resistance, which means that acceleration is a constant $-g$, velocity is linear, and position is quadratic. To be precise, we have

$$v = -9.8t + 20,$$

$$x = -4.9t^2 + 20t + 30.$$

Here are graphs of all four curves. Velocity in blue, position in red; the ones with air resistance are solid, and the ones without are dotted.



As you can see, the ball thrown without air resistance makes it 5 meters or so higher than the one with air resistance, but they hit the ground around the same time.

22. *The setup is the same as the previous problem except that there is a force due to air resistance of magnitude $v^2/1325$ directed opposite to the velocity, where the velocity v is measured in m/s.* [5 pts, and the math is pretty hairy, so go easy on them. Some of them may have numerically integrated at some point, which is all right.]

(a) *Find the maximum height above the ground the ball reaches.* The right model is now

$$v' = \begin{cases} -g - \frac{v^2}{1325m} & v \geq 0 \\ -g + \frac{v^2}{1325m} & v < 0. \end{cases} \tag{4}$$

Since v^2 is always positive, we need to change the sign to make sure that drag points opposite to velocity. Unfortunately, this means that the differential equation has to be solved piecewise. To solve part (a), we just need the part where $v \geq 0$.

We solve the equation:

$$v' = -g - \frac{v^2}{198.75}$$
$$\int \frac{1}{1 + v^2/198.75g}\, dv = \int -g\, dt.$$

Here, let's do the u-substitution $u = v/\sqrt{198.75g}$. Then $du = dv/\sqrt{198.75g}$, so $dv = \sqrt{198.75g}\, du$. We get

$$\begin{aligned}
\int \frac{\sqrt{198.75g}}{1+u^2}\, du &= \int -g\, dt \\
\sqrt{198.75g}\tan^{-1}(u) &= -gt + C \\
\sqrt{198.75g}\tan^{-1}(v/\sqrt{198.75g}) &= -gt + C \\
\tan^{-1}(v/\sqrt{198.75g}) &= -\sqrt{\frac{g}{198.75}}t + C \\
v = \sqrt{198.75g}\tan\left(C - \sqrt{\frac{g}{198.75}}t\right) &= 44.133\tan(C - 0.22205t)
\end{aligned}$$

Since $v(0) = 20$, we have $20 = 44.133\tan(C)$, or $C = 0.42549$. The equation for v is thus

$$v = 44.133\tan(0.42549 - 0.22205t). \tag{5}$$

Aside. Three things worth pointing out here. First, the tangent function takes input in radians, not degrees, which will matter in a second when we compute arctan. This is just because it's the radian arctangent function $\tan^{-1}(u)$ whose derivative is $1/(1+u^2)$. Second, the tangent function has an asymptote at $-\pi/2$, so that v goes to $-\infty$ at $t \approx 9$ s. This is a weird thing for velocity to do, but it turns out not to matter: v hits zero well before this, at which point the sign on the drag force changes and a different equation governs v. Third, let's check units. For $v^2/198.75$ to be an acceleration [m/s^2], 198.75 must be a length [m]. This means that $\sqrt{g/198.75}$ has units of s^{-1}, so that $\sqrt{g/198.75}t$, and thus C, are dimensionless, which we expect as the input of a function like tan. The $\sqrt{198.75g}$ on the outside has the expected units of m/s.

Now, $v = 0$ when $0.42549 - 0.22205t = 0$, or $t = 1.9162$s. This is the peak of the curve, and the differential equation we solved stops being valid here. However, we want the *distance* that the object travels, not the *time*.

To get a formula for distance y, we integrate (5).

$$\begin{aligned}
y = \int v\, dt = \int 44.133\tan(0.42549 - 0.22205t)\, dt \\
&= 44.133 \int \frac{\sin(0.42549 - 0.22205t)}{\cos(0.42549 - 0.22205t)}\, dt \\
&= \frac{44.133}{0.22205}\ln(\cos(0.42549 - 0.22205t)) + C.
\end{aligned}$$

Note that the input to cosine is between 0 and $\pi/2$ in the domain we're considering, so its cosine will always be nonnegative, meaning we don't need an absolute value. The initial condition $y(0) = 30$ leads to the solution

$$y = 198.75\ln(\cos(0.42549 - 0.22205t)) + 48.562\,[\text{m}]. \tag{6}$$

When $t = 1.9162$, $y = 48.562\,\text{m}$.

(b) *Find the time that the ball hits the ground.*

We now need to solve the *other* case of the original differential equation (4) – what we have so far is only valid when $v \geq 0$, i. e. up to the peak.

We follow the same process as with the first case.

$$\begin{aligned}
v' &= -g + \frac{v^2}{198.75} \\
\int \frac{1}{1 - v^2/198.75g}\, dv &= \int -g\, dt. \quad \text{Let } u = v/\sqrt{198.75g}\ldots \\
\sqrt{198.75g} \int \frac{1}{1-u^2}\, du &= -gt + C. \quad \text{Use partial fractions}\ldots \\
\frac{\sqrt{198.75g}}{2} \int \left(\frac{1}{1+u} + \frac{1}{1-u} \right) du &= -gt + C \\
\frac{\sqrt{198.75g}}{2} \left(\ln(1 + v/\sqrt{198.75g}) - \ln(1 - v/\sqrt{198.75g}) \right) &= -gt + C \\
\ln \left(\frac{44.133 + v}{44.133 - v} \right) &= -0.44411t + C.
\end{aligned}$$

We should solve for C here – what is the initial condition? Since we're on the way down, $v = 0$ at the peak, i. e. at $t = 1.9162$s. It would be fine to call this $t = 0$ as well as long as we remember that we'd relabelled time from the previous part of the problem, and added the time we get at the end to 1.9162 s. We get $C = 0.44411 \times 1.9162 = 0.85100$. Now we solve for v in terms of t.

$$\begin{aligned}
\frac{44.133 + v}{44.133 - v} &= e^{-0.44411t+0.85100} \\
44.133 + v &= 44.133e^{-0.44411t+0.85100} - ve^{-0.44411t+0.85100} \\
v &= 44.133 \frac{e^{-0.44411t+0.85100} - 1}{e^{-0.44411t+0.85100} + 1}.
\end{aligned}$$

It's worth noticing that this approaches a terminal velocity of -44.133 m/s, which is reasonable.

It remains to integrate again to get the time at which $y = 0$. It would be fine to do this numerically, since the formula for v is complicated. As explained in class, there's a trick you can use to integrate v. Multiply top and bottom of the fraction by $e^{0.22205t}$ to get

$$v = 44.133 \frac{e^{-0.22205t+0.85100} - e^{0.22205t}}{e^{-0.22205t+0.85100} + e^{0.22205t}}.$$

Now the derivative of the denominator is -0.22205 times the top. Thus, the integral of v is

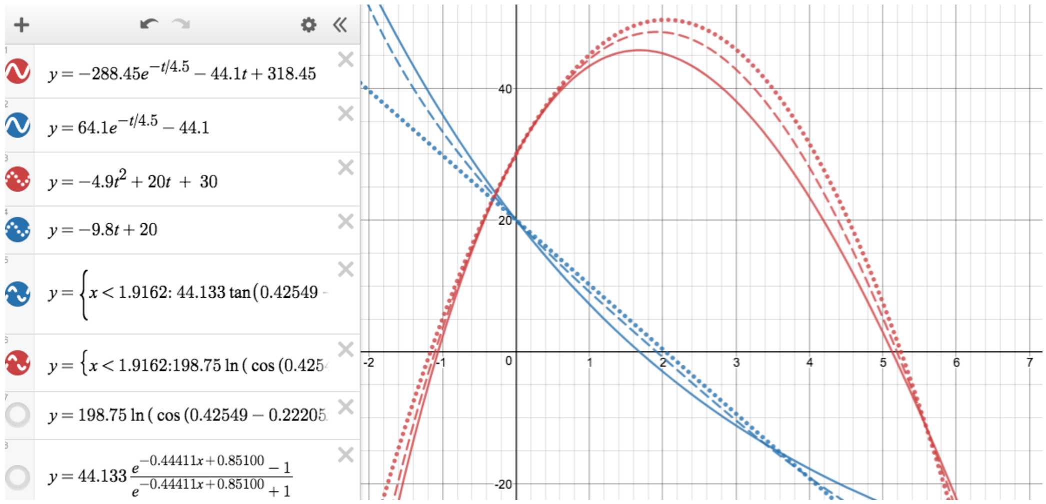
$$\begin{aligned}
y &= \frac{44.133}{-0.22205} \ln(e^{-0.22205t+0.85100} + e^{0.22205t}) + C \\
&= -198.75 \ln(e^{-0.22205t+0.85100} + e^{0.22205t}) + C.
\end{aligned}$$

Since $y(1.9162) = 48.562$ m, we get $C = 270.89$ m. Thus,

$$y = -198.75 \ln(e^{-0.22205t+0.85100} + e^{0.22205t}) + 270.89,$$

which reaches zero at $t = 5.194$s.

(c) *Plot the graphs of velocity and position versus time. Compare these graphs with the corresponding ones in Problems 20 and 21.*



On this graph, velocity is in blue and position is in red. The solid, dashed, and dotted curves correspond respectively to linear, quadratic, and no air resistance. We see that quadratic air resistance allows the ball to rise slightly higher than linear air resistance, but not as high as no air resistance.

To graph the curves from this problem, I used Desmos's piecewise capability. For example, the equation I entered for the velocity curve is

$$y = \left\{ x < 1.9162 : 44.133 \tan\left(0.42549 - 0.22205x\right), x \geq 1.9162 : 44.133 \frac{e^{-0.44411x+0.85100} - 1}{e^{-0.44411x+0.85100} + 1} \right\}.$$

You can play with the graph online at `https://www.desmos.com/calculator/k17vhu4hxz`.

29. *Suppose that a rocket is launched straight up from the surface of the earth with initial velocity $v_0 = \sqrt{2gR}$, where R is the radius of the earth. Neglect air resistance.* [5 pts]

This problem is related to example 4 in the text. There, it's explained that $\sqrt{2gR}$ is the *escape velocity* of the rocket – the minimum initial velocity at which it can be launched so that it escapes the earth's gravitational pull (i. e., its distance from earth approaches infinity.) It's also explained that, to approach this type of problem, you can't model acceleration due to gravity as a constant g, but you have to use Newton's law of gravitation, which in this case reduces to

$$v' = -\frac{gR^2}{(R+x)^2}. \tag{7}$$

It is all right to use some of the work done in example 4 in this problem.

(a) *Find an expression for the velocity v in terms of the distance x from the surface of the earth.*

As explained in the text,

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

So we may rewrite (7) as

$$v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2},$$

which is separable. Integrating gives

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + C,$$

or

$$v = \sqrt{\frac{2gR^2}{R+x} + C}.$$

We must choose the positive square root because the rocket is going up, so its velocity is positive. Since $v(0) = \sqrt{2gR}$, we have $C = 0$. The solution is

$$v = \sqrt{\frac{2gR^2}{R+x}}. \tag{8}$$

(b) *Find the time required for the rocket to go 240,000 mi (the approximate distance from the earth to the moon). Assume that* $R = 4000\,\mathrm{mi}$.

The trick is that we can now treat (8) as a differential equation for x. The assumption that the rocket launches at escape velocity makes this differential equation significantly easier to solve (compare what would happen if $C \neq 0$ in the previous part.) We have

$$\begin{aligned} x' &= \sqrt{\frac{2gR^2}{R+x}} \\ \int (R+x)^{1/2}\,dx &= \int \sqrt{2gR^2}\,dt \\ \frac{2}{3}(R+x)^{3/2} &= \sqrt{2gR^2}t + C. \end{aligned}$$

We can put $x = 0$ at $t = 0$, so that $C = \frac{2}{3}R^{3/2}$. Also, we have

$$g = 32\,\mathrm{ft/s^2} = 32/5280\,\mathrm{mi/s^2} = 32 \times (3600)^2/5280\,\mathrm{mi/hr^2} \approx 79000\,\mathrm{mi/hr^2}.$$

So, in numbers, the above is

$$\frac{2}{3}(4000+x)^{3/2} = (1.6 \times 10^6)t + 170000$$

where t is in hours and x is in miles. When $x = 240000$, we obtain

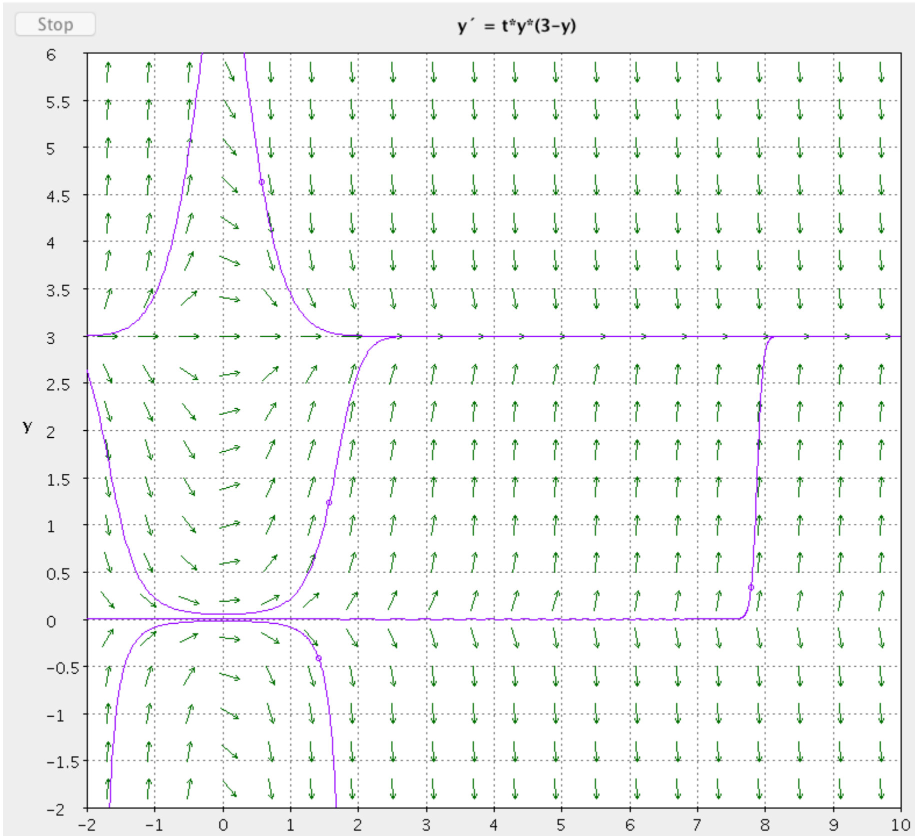
$$t \approx 50\,\mathrm{hr}.$$

ASSIGNMENT 8.

2.4.17. [6 pts – they should have all the cases below.] *Draw a direction field and plot (or sketch) several solutions of the differential equation. Describe how solutions appear to behave as* t *increases and how their behavior depends on the initial value* y_0 *when* $t = 0$.

$$y' = ty(3-y)$$

Here is a direction field:



You should have noticed the following.

- There are constant solutions at $y = 0$ and $y = 3$.
- If $y_0 < 0$, solutions diverge to $-\infty$.
- If $0 < y_0 < 3$ or $y_0 > 3$, solutions converge to 3.

2.4.22(a,b). (a) [3 pts] *Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem*

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \quad y(2) = -1.$$

Where are these solutions valid?

If $y_1(t) = 1 - t$, then $y_1' = -1$. We calculate

$$\frac{-t + \sqrt{t^2 + 4y}}{2} = \frac{-t + \sqrt{t^2 - 4t + 4}}{2} = \frac{-t + (t - 2)}{2} = -1.$$

If $y_2(t) = -t^2/4$, then $y_2' = -t/2$. We calculate

$$\frac{-t + \sqrt{t^2 + 4y}}{2} = \frac{-t}{2}.$$

Both functions are continuously differentiable everywhere. Also, the calculations we made are valid everywhere: at no point did we have to take the square root of a negative number, or divide by zero, or the like. So the solutions are valid everywhere.

(b) [5 pts – I give two things that go wrong, but either one is fine.] *Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.*

Let $f(t, y) = \frac{-t+\sqrt{t^2+4y}}{2}$. Note that f is not defined precisely when

$$y < -t^2/4.$$

In particular, there is no open rectangular region around the point $(2, -1)$ on which f is continuous. The existence of such a region is one of the hypotheses of the theorem.

Moreover,

$$\frac{\partial f}{\partial y} = (t^2 + 4y)^{-1/2},$$

which diverges at $(2, -1)$. The continuity of this function of y and t in some open rectangular region around $(2, -1)$ is another hypothesis of the theorem.

So the theorem just doesn't apply here.

Aside. Making a graph exposes this problem's Hellenic flavor. See `https://www.desmos.com/calculator/jdlaqxf6up` for mine. The parabola $y_2 = -t^2/4$ is exactly on the boundary of the region where f is not defined, i. e., where the differential equation isn't solvable. The line y_1 is tangent to this parabola at the point $(2, -1)$ – which you already knew, because this is just saying that they both pass through this point and have the same derivative. As the textbook goes on to point out, there is a linear 'general solution' to the equation of the form $y = Ct + C^2$ (putting $C = -1$ gives back y_1). Letting C vary, we see that these are *all* the tangent lines to the parabola. So each instance of the 'general solution' touches the forbidden region where the equation isn't solvable, and the points at which they touch it themselves form a curve that also solves the equation.

D. [6 pts] *Find the explicit solution of the Separable Equation*

$$\frac{dy}{dt} = y^2 - 4y, y(0) = 8.$$

What is the largest open interval containing $t = 0$ *for which the solution is defined?*

Separating the variables gives

$$\frac{1}{y(y-4)}\,dy = dt.$$

We can handle the left-hand side with partial fractions, which gives

$$\frac{1}{4}\left(\frac{1}{y-4} - \frac{1}{y}\right) dy = dt.$$

Integrating gives

$$\begin{aligned} \frac{1}{4}\ln\left|\frac{y-4}{y}\right| &= t + C \\ \left|\frac{y-4}{y}\right| &= Ae^{4t}. \end{aligned}$$

From the initial condition, we get $A = \frac{1}{2}$. We also notice (from dfield, or just familiarity with this kind of equation) that if $y(0) > 4$, y will never be less than 4. So we can discard the absolute values, and solve for y:

$$\begin{aligned} y - 4 &= \frac{1}{2}e^{4t}y \\ y &= \frac{4}{1 - e^{4t}/2} \\ y &= \frac{8}{2 - e^{4t}}. \end{aligned}$$

The solution is defined as long as $e^{4t} < 2$ – that is, on the open interval $(-\infty, \ln(2)/4)$. As in Example 4 in section 2.4 (the equation $y' = y^2$, which we discussed in class), varying the initial condition $y(0)$ will vary the value of t at which this vertical asymptote appears.