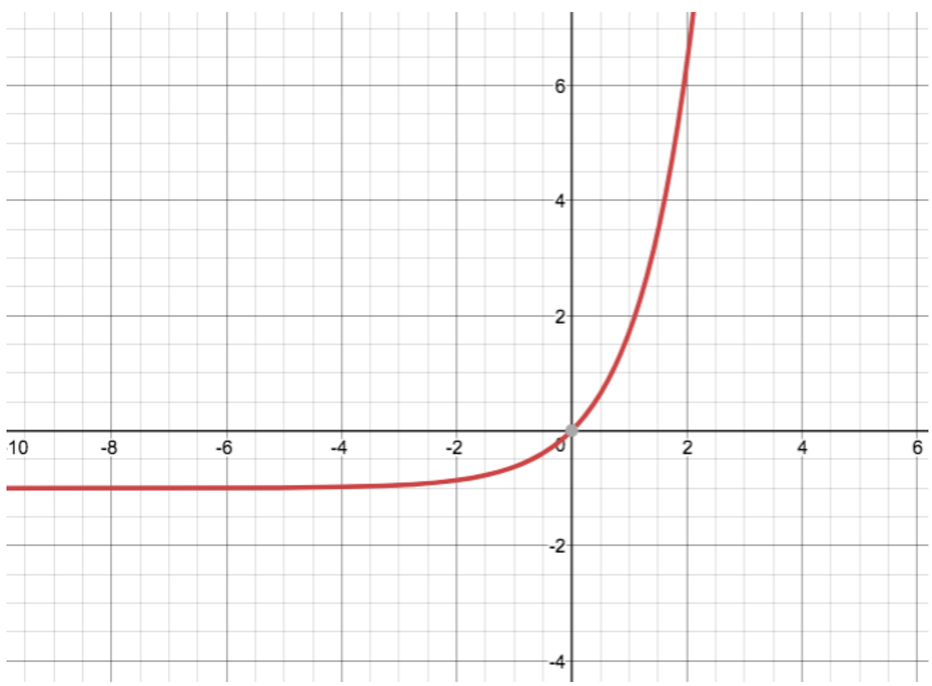
Solutions to Week 4 Homework

ASSIGNMENT 9.

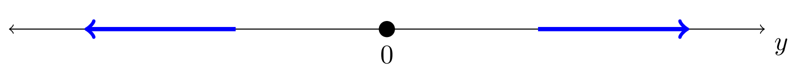
2.5.4. *In the following equation $dy/dt = f(y)$, sketch the graph of $f(y)$ versus y, determine the critical (equilibrium) points, and classify each one as asymptotically stable or unstable. Draw the phase line, and sketch several graphs of solutions in the ty-plane.*

$$dy/dt = e^y - 1, \quad -\infty < y_0 < \infty.$$

Here is a graph of $f(y) = e^y - 1$:

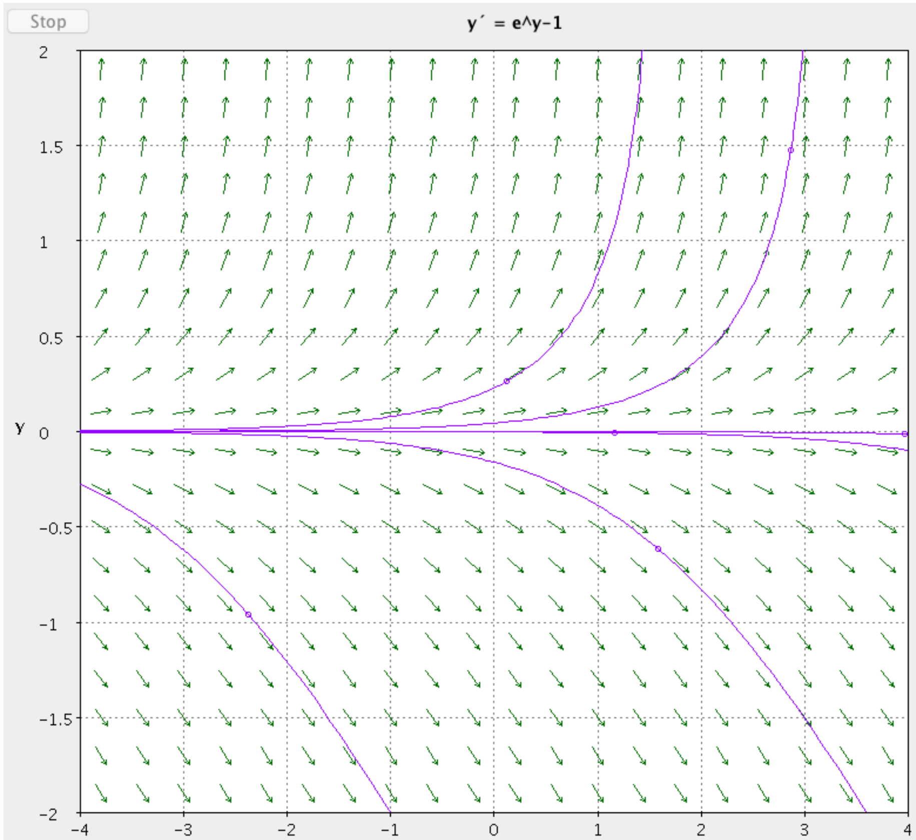


The only zero is at $y = 0$. The function is positive for positive y, and negative as negative y. This gives the following phase line:



There is only one equilibrium solution, namely $y = 0$. Since nearby solutions are pushed away from it – y' is positive for $y > 0$ and negative for $y < 0$ – it is an unstable equilibrium.

Here are some solutions:



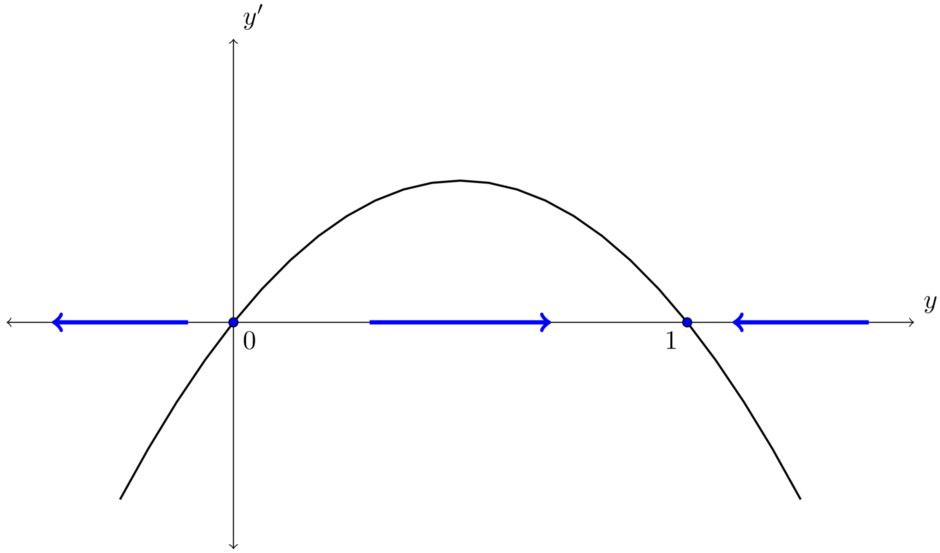
22. *Suppose that a given population can be divided into two parts: those who have a given disease and can infect others, and those who do not have it but are susceptible. Let x be the proportion of susceptible individuals and y the proportion of infectious individuals; then $x+y=1$. Assume that the disease spreads by contact between sick and well members of the population and that the rate of spread dy/dt is proportional to the number of such contacts. Further, assume that members of both groups move about freely among each other, so the number of contacts is proportional to the product of x and y. Since $x = 1-y$, we obtain the initial value problem*

$$dy/dt = \alpha y(1-y), \quad y(0) = y_0, \tag{1}$$

where α is a positive proportionality factor, and y_0 is the initial proportion of infectious individuals.

(a) *Find the equilibrium points for the differential equation* (1) *and determine whether each is asymptotically stable, semistable, or unstable.*

The graph of y' as a function of y, and the phase line, look something like this. (It's important that α is positive – if it were negative, we'd have very different behavior).



We can see that there are two equilibrium solutions: $y = 0$, which is unstable, and $y = 1$, which is stable.

(b) *Solve the initial value problem* (1) *and verify that the conclusions you reached in part (a) are correct. Show that* $y(t) \to 1$ *as* $t \to \infty$*, which means that ultimately the disease spreads through the entire population.*

The equation is separable, so we can write

$$\frac{1}{y(1-y)}\, dy = \alpha\, dt,$$

and use partial fractions to split up the left-hand side into $\left(\frac{1}{y} + \frac{1}{1-y}\right) dy$. We obtain

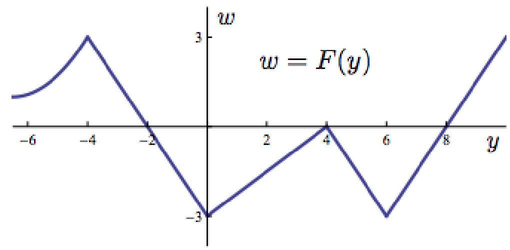
$$\begin{aligned}
\int \left(\frac{1}{y} - \frac{1}{1-y}\right) dy &= \int \alpha\, dt \\
\ln|y| - \ln|1-y| &= \alpha t + C \\
\left|\frac{y}{1-y}\right| &= Ae^{\alpha t} \quad \text{Relabel } \pm A \text{ as } A\ldots \\
\frac{y}{1-y} &= Ae^{\alpha t} \\
y &= Ae^{\alpha t} - Ae^{\alpha t} y \\
y &= \frac{Ae^{\alpha t}}{1 + Ae^{\alpha t}} \\
y &= \frac{1}{1 + Ae^{-\alpha t}}.
\end{aligned}$$

The initial value $y(0) = y_0$ turns into $y_0 = \frac{1}{1+A}$, or $A = \frac{1-y_0}{y_0}$. The solution simplifies to

$$y = \frac{y_0}{y_0 + (1-y_0)e^{-\alpha t}}. \tag{2}$$

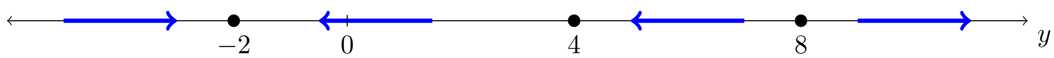
As $t \to \infty$, since $\alpha > 0$, the exponential term goes to zero, and y goes to 1. Observe that the initial values $y_0 = 0$ and $y_0 = 1$ do give you the expected constant solutions when you substitute them into (2).

E. *The graph of* $F(y)$ *versus* y *is as shown:*



(a) *Find the equilibrium solutions of the autonomous differential equation* $\frac{dy}{dt} = F(y)$.
There is an equilibrium solution $y = a$ whenever $F(a) = 0$. So the equilibrium solutions are at $y = -2$, 4, and 8.

(b) *Determine the stability of each equilibrium solution.*
Here is the phase line:



Remember that we get this by looking at the sign of y' (that is, $F(y)$) depending on y. A positive sign for F means that $y' > 0$ at this value of y – solutions at this value of y are pushed in the positive direction with increasing time. Likewise, a negative sign for F means that solutions are pushed in the negative direction.
Looking at the phase line, we see that $y = -2$ is stable, $y = 4$ is semistable, and $y = 8$ is unstable.

ASSIGNMENT 10.

2.6.18. *Show that any separable equation*

$$M(x) + N(y)y' = 0$$

is also exact.

Recall that the criterion for exactness for an equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If M is only a function of x and N is only a function of y, then both sides are zero. So such an equation is exact.

To solve it "as an exact equation", we need to find a function $\psi(x,y)$ such that $\partial\psi/\partial x = M$ and $\partial\psi/\partial y = N$. Such a function is given by

$$\psi(x,y) = \int M(x)\,dx + \int N(y)\,dy.$$

This has secret constants of integration in it, since the integrals are indefinite, so to get a specific ψ, we'd really want to write

$$\psi(x,y) = \int_{x_0}^{x} M(s)\,ds + \int_{y_0}^{y} N(t)\,dt,$$

where the values x_0 and y_0 are chosen so that these integrals are actually defined. The general solution is then

$$\int_{x_0}^{x} M(s)\,ds + \int_{y_0}^{y} N(t)\,dt = C.$$

You should check that this is the same as what you'd get if you separated the variables.

There's a liltle more going on here – the exact equation theorem only applies in an open rectangular region of the xy-plane where M, N, $\partial M/\partial y$, and $\partial N/\partial x$ are all continuous. In this case, $\partial M/\partial y = \partial N/\partial x = 0$ is continuous everywhere. Meanwhile, M and N are functions of 1 variable, so if, for example, M has an isolated discontinuity at $x = a$, then viewed as a function of two variables, it will be discontinuous along the whole line $x = a$ in the xy-plane. Likewise, isolated discontinuities of N appear as discontinuities along lines of the for $y = b$. All the functions we care about in this class have isolated discontinuities (or are undefined on a whole half-line, like the natural logarithm), so we expect to be able to find an open rectangular region where M and N are both continuous, and use the exact equation theorem there.

Of course, we need this sort of assumption to solve a separable equation, anyway. The ordinary way to solve

$$M(x) + N(y)\frac{dy}{dx} = 0$$

is to separate the variables and integrate:

$$\int N(y)\,dy = \int -M(x)\,dx.$$

Typically, we need to assume that the functions M and N are continuous to do these integrals.

F. *Solve the diffferential equation*

$$\frac{dw}{dt} = \frac{2tw}{w^2 - t^2}.$$

Let's write this as

$$-2tw + (w^2 - t^2)\frac{dw}{dt} = 0. \tag{3}$$

As you probably guessed, this is exact. Let's check by applying the exact equation theorem. The functions $-2tw$ and $(w^2 - t^2)$ are continuous and have continuous derivatives, so the theorem applies everywhere. The derivative of $-2tw$ with respect to w is $-2t$, which is the same as the derivative of $(w^2 - t^2)$ with respect to t. Thus, the equation is exact.

It's important to compare the correct derivatives here! You can figure out which ones you're supposed to do by repeating the argument from class using the multivariable chain rule. If w is a function of t, then

$$\frac{d}{dt}(\psi(t,w)) = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial w}\frac{dw}{dt}.$$

Thus, for the left side of (3) to be of this form, we need to have

$$\frac{\partial \psi}{\partial t} = -2tw, \quad \frac{\partial \psi}{\partial w} = w^2 - t^2. \tag{4}$$

This forces

$$\frac{\partial}{\partial w}(-2tw) = \frac{\partial^2 \psi}{\partial w \partial t} = \frac{\partial^2 \psi}{\partial t \partial w} = \frac{\partial}{\partial t}(w^2 - t^2).$$

So this is the equality we have to check to apply the exact equation theorem. Another way to keep track of this is just to rename w and t to y and x and use the version you learned.

Anyway, we need to find ψ satisfying (4). A ψ satisfying the first equation is $-t^2 w$. The partial derivative of this with respect to w is $-t^2$. We can correct this by adding an antiderivative of w^2 with respect to w, such as $w^3/3$. We get

$$\psi(t,w) = w^3/3 - t^2.$$

So the general solution is

$$w^3/3 - t^2 = C.$$