Solutions to Week 5 Homework

ASSIGNMENT 11.

2.7.20. *Consider the initial value problem*

$$y' = 1 - t + y, \quad y(t_0) = y_0.$$

We will show that the approximation generated by the Euler method converges to the actual solution as the step size h decreases.

(a) *Show that the exact solution is*

$$y = \phi(t) = (y_0 - t_0)e^{t-t_0} + t.$$

This is a linear differential equation, so we can solve it using the integrating factor method. We rewrite it as

$$y' - y = 1 - t,$$

so the integrating factor is

$$\mu(t) = \exp\left(\int -1\, dt\right) = e^{-t}.$$

We compute:

$$\begin{aligned} e^{-t}y' - e^{-t}y &= e^{-t}(1-t) \\ (e^{-t}y)' &= e^{-t}(1-t) \\ e^{-t}y &= \int (1-t)e^{-t}\, dt = \int e^{-t}\, dt - \int te^{-t}\, dt \\ &= -e^{-t} - \int te^{-t}\, dt. \end{aligned}$$

This last integral can be computed by parts. Taking $u = t$ and $dv = e^{-t}\, dt$, we have $du = dt$ and $v = -e^{-t}$, so that

$$\int te^{-t}\, dt = -te^{-t} + \int e^{-t}\, dt = (-1-t)e^{-t} + C.$$

Thus,

$$e^{-t}y = te^{-t} + C,$$

giving the general solution

$$y = t + Ce^{t}.$$

Plugging in the initial condition $y(t_0) = y_0$, we obtain

$$y_0 = t_0 + Ce^{t_0}$$

or

$$C = (y_0 - t_0)e^{-t_0}.$$

This gives the solution to the equation:

$$y = \phi(t) = (y_0 - t_0)e^{t-t_0} + t.$$

(b) *Using the Euler formula, show that*

$$y_k = (1+h)y_{k-1} + h - ht_{k-1}, \quad k = 1, 2, \ldots . \tag{1}$$

Here, (t_k, y_k) is the kth point obtained by Euler's method with a step size of h. Euler's method says that, if the differential equation is $y' = f(t, y)$,

$$y_k = f(t_{k-1}, y_{k-1})h + y_{k-1}.$$

In this case, this means that

$$y_k = (1 - t_{k-1} + y_{k-1})h + y_{k-1} = (1+h)y_{k-1} + h - ht_{k-1}.$$

(d) *Consider a fixed point $t > t_0$ and for a given n choose $h = (t - t_0)/n$. Then $t_n = t$ for every n. Note also that $h \to 0$ as $n \to \infty$. By substituting for h in* (1) *and letting $n \to \infty$, show that $y_n \to \phi(t)$ as $n \to \infty$.*
Hint: $\lim_{n\to\infty}(1 + a/n)^n = e^a$.
We assume the result from part (c) of this problem, that for any n,

$$y_n = (1+h)^n(y_0 - t_0) + t_n.$$

Since we have chosen $h = (t - t_0)/n$, so that $t_n = t$, this is equivalent to

$$y_n = \left(1 + \frac{t - t_0}{n}\right)^n (y_0 - t_0) + t.$$

Using the hint, we get

$$\lim_{n\to\infty} y_n = e^{t-t_0}(y_0 - t_0) + t.$$

But this is just the formula for $\phi(t)$ obtained in part (a).

ASSIGNMENT 12.

G. (a) *If $y' = -2y + e^{-t}$, $y(0) = 1$ then compute $y(1)$.*
The equation is linear, so we can solve it with the integrating factor method. Writing

$$y' + 2y = e^{-t},$$

we get an integrating factor of $\mu(t) = \exp\left(\int 2\,dt\right) = e^{2t}$. The equation becomes

$$\begin{aligned} (e^{2t}y)' &= e^{2t-t} = e^t \\ e^{2t}y &= \int e^t\,dt = e^t + C \\ y &= e^{-t} + Ce^{-2t}. \end{aligned}$$

Since $y(0) = 1$, C must equal 0. Thus, $y(t) = e^{-t}$ and

$$y(1) = e^{-1} = 0.368.$$

(Since the next part of the problem asks about an accuracy of 0.05, three decimal places are all that we'll need.

(b) *Experiment using the Euler Method* (eul) *with step sizes of the form $h = 1/n$ to find the smallest integer n which will give a value y_n that approximates the above true solution at $t = 1$ within* 0.05.

I used the following MATLAB code, saved as problemG.m:

```
function f = problemG(t,y)
     f = -2*y + exp(-t);
end
```

and in the command line:

```
[t,y] = eul('problemG', [0,1], 1, 1/2); [t,y]
```

with 1/2 replaced by different choices of h.

When the step size is 1/2, eul returns a final y-value of 0.303. When the step size is 1/3, eul returns a final y-value of 0.325, which is within 0.05 of the true value. So the smallest value of n is 3.

Instead of choosing n by hand, one could also, example, write a for loop that tries different values of n until it gets close enough to the real answer.

H. (a) *If $y' = 2y - 3e^{-t}$, $y(0) = 1$ then compute $y(1)$.*

Again, we write

$$y' - 2y = -3e^{-t}$$

and multiply by the integrating factor e^{-2t} to get

$$e^{-2t}y = \int -3e^{-3t}\,dt = e^{-3t} + C$$

or

$$y = e^{-t} + Ce^{2t}.$$

Since $y(0) = 1$, we again have $C = 0$, so $y = e^{-t}$ and $y(1) = 1/e \approx 0.368$.

(b) *Experiment using the Euler Method (eul) with step sizes of the form $h = 1/n$ to find the smallest integer n which will give a value y_n that approximates the above true solution at $t = 1$ within 0.05.*

I used the following MATLAB code, saved as problemH.m:

```
function f = problemH(t,y)
     f = 2*y  - 3*exp(-t);
end
```

and in the command line:

```
[t,y] = eul('problemH', [0,1], 1, 1/2); [t,y]
```

I found that $h = 1/22$ was the first to get within 0.05 of the correct answer, returning $y_{22} = 0.320$.

The point of this and the last problem is hopefully clear: the rate of convergence of the Euler method depends on the differential equation, not just its solution.

I. (a) *Show that $y(t) = \int_0^t e^{-u^2}\,du$ satisfies the initial value problem $\frac{dy}{dt} = e^{-t^2}$, $y(0) = 0$.*

This is the fundamental theorem of calculus. If $F(t) = \int_a^t f(u)\,du$ for some constant a, and f is continuous on (a, b), then $\frac{d}{dt}F(t) = f(t)$ for $t \in (a, b)$. In this case, we have

$$\frac{d}{dt}\int_0^t e^{-u^2}\,du = e^{-t^2},$$

as desired. Also,

$$y(0) = \int_0^0 e^{-u^2}\,du = 0.$$

Don't get confused here. y is a function of t, not of u – u is only used to define the integration. So you shouldn't have had to 'change variables' from u to t or the like.

(b) *Use the Euler method (*eul*) with $h = 1/2$ to approximate the integral $\int_0^2 e^{-u^2}\,du$.* The point is to treat the integral as the solution to the above initial value problem, and use the Euler method on that. My problemI.m was:

```
function f = problemI(t,y)
     f = exp(-t^2);
end
```

I then wrote

```
[t,y] = eul('problemI', [0,1], 1, 1/2); [t,y]
```

and got an approximate value of $y(1) \approx 1.126$.

As an additional exercise, you should (a) figure out how to generalize this method to approximate any definite integral whatsoever; (b) show that this Euler method approximation with a step size h is *the same* as the approximation of the integral by a Riemann sum with box width h.